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THE ENERGY PRINCIPLE APPLIED TO DIVERTED TOKAMAK CONFIGURATIONS*

C. V. ATANASIU[1], S. GÜNTER[2], K. LACKNER[2], A. MORARU[3], L. E. ZAKHAROV[4], A. A. SUBBOTIN[5]

[1] *National Institute for Laser Plasma and Radiation Physics, MEdC-EURATOM Association, P.O.Box MG-36, Bucharest, Romania, E-mail: cva@ipp.mpg.de*
[2] *Max-Planck-Institut für Plasmaphysik, EURATOM Association, Garching, Germany,*
[3] *Princeton University, Plasma Physics Laboratory, Princeton, New Jersey 08543, USA,*
[4] *Polytechnical University of Bucharest, R-77206 Bucharest, Romania,*
[5] *Russian Research Center, I.V. Kurchatov Institute, 12382 Moscow, Russian Federation*

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Abstract. Writing the expression of the potential energy in terms of the perturbation of the flux function, and performing an Euler minimisation, one obtains a system of ordinary differential equations in that perturbation. For a diverted configuration, the usual vanishing boundary conditions for the perturbed flux function at the magnetic axis and at infinity can no longer be used. In place of the vanishing boundary conditions at infinity, an approach to fix "natural" boundary conditions for the system of differential equations for the perturbed flux function, just at the plasma boundary has been developed.

Key words: MHD stablity, fusion, tokamak.

1. INTRODUCTION

Modes having a resistivity sufficiently small such that the Lundquist number is at least, say 10^6, can be described by an asymptotic matching method. This method consists of dividing the plasma into an outer region, where resistivity and growth rates are negligible, and inner layers of small width containing the rational surfaces. The outer region and the inner layers give rise to solutions to be matched.

Classical linear resistive mode theory [1] predicts stability when the linear stability index for the tearing mode Δ', defined as the jump of the logarithmic derivative of the flux function perturbation across the rational surface, is negative for the pressureless limit, or smaller than a positive threshold in the finite pressure

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case if the resistive interchange index is less then zero (and unstable if this index is positive) [2]. There are different approaches to solve the linear stability problem [3–15].

In the following, based manly on our previous contribution [16] we present an extension of our method to calculate the linear stage of the tearing modes in a diverted geometry. Our solving method is very fast and stable even for modes close to marginal stability. The form in which the tearing modes equations have been written is suitable to investigate the stabilization of these modes by current drive. We intend to pay attention mainly to the special aspects due to the presence of the separatrix for tearing modes calculations with application to a specific discharge of the ASDEX Upgrade (Axially Symmetric Divertor EXperiment) tokamak [17].

2. GENERAL EXPRESSION OF THE POTENTIAL ENERGY

In a coordinate system with straight field lines (a, θ, ζ), the equilibrium contravariant components of the magnetic field B^i and of the current density j^i are

$$B^i = \left\{0,\ -\frac{\Psi'(a)}{2\pi\sqrt{g^r}}, \frac{\Phi'(a)}{2\pi\sqrt{g^r}}\right\}, \qquad j^i = \left\{0,\ -\frac{F'(a)}{2\pi\sqrt{g^r}}, \frac{J'(a)+\partial\nu/\partial\theta}{2\pi\sqrt{g^r}}\right\}. \tag{1}$$

The superscript ^{r} indicates that the metric coefficients are defined in the straight field lines coordinate system. In the same coordinate system, considering a symmetry with respect to the ζ coordinate, *i.e.* $\partial/\partial\zeta \equiv 0$, Maxwell and equilibrium equations are

$$\mu_0\left(J' + \frac{\partial\nu}{\partial\theta}\right) = -\frac{\partial}{\partial a}\left(\frac{g_{22}^r}{\sqrt{g^r}}\Psi'\right) + \frac{\partial}{\partial\theta}\left(\frac{g_{12}^r}{\sqrt{g^r}}\Psi'\right), \tag{2}$$

$$4\pi^2\sqrt{g^r}\,p' = -F'\Phi' + \left(J' + \frac{\partial\nu}{\partial\theta}\right)\Psi', \tag{3}$$

where $p(a)$ is the plasma pressure, $\chi(a) = \Psi(0) - \Psi(a) = \int_0^a da \int_0^{2\pi} B^2\sqrt{g^r}\,d\zeta$ is the poloidal magnetic flux, $\Phi(a) = \int_0^a da \int_0^{2\pi} B^3\sqrt{g^r}\,d\theta$ is the toroidal magnetic flux, $J'(a) + \partial\nu/\partial\theta$ is the toroidal current, $F(a) = g_{33}^r/(\sqrt{g^r}\,\mu_0)\Phi'$ is the poloidal current, g_{ik}^r is the metric tensor, $\sqrt{g^r} = \left|\nabla a\cdot[\nabla\theta\times\nabla\zeta]\right|^{-1}$ is the Jacobian of the coordinate system, while ν is, in general, a periodic function of θ and ζ. The prime symbol denotes differentiation with respect to a, and will be used throughout this paper. The differentiations with respect to θ and ζ will be noted with $(...)'_\theta$ and $(...)'_\zeta$, respectively.

The energy integral W [19] due to a displacement $\vec{\xi}$ of the plasma from the equilibrium position is given by the well known relation

$$W = \frac{1}{2\mu_0}\int [\tilde{\mathbf{B}}^2 - \vec{\xi}\times(\nabla\times\mathbf{B})\cdot\tilde{\mathbf{B}} + \mu_0(\vec{\xi}\cdot\nabla p)(\nabla\cdot\vec{\xi}) + \mu_0\gamma p(\nabla\cdot\vec{\xi})^2]dV, \tag{4}$$

where the displacement vector has the components $\{\xi^a, \xi^\theta, \xi^\zeta\}$, and the first order perturbed magnetic field $\tilde{\mathbf{B}}$ is given by

$$\tilde{\mathbf{B}} = \nabla\times(\vec{\xi}\times\mathbf{B}), \tag{5}$$

γ is the adiabatic constant.

Introducing two test functions $u(a,\theta,\zeta)$ and $\lambda(a,\theta,\zeta)$ by help of the relations [20, 21]

$$\Phi'\xi^\theta + \Psi'\xi^\zeta \equiv \lambda - \frac{\partial u}{\partial a}, \tag{6}$$

$$\Phi'\xi^a \equiv \frac{\partial u}{\partial \theta}, \tag{7}$$

$$\psi \equiv -\frac{\Psi'}{\Phi'}\frac{\partial u}{\partial\theta} + \frac{\partial u}{\partial\zeta} = -\Psi'\xi^a + \frac{\partial u}{\partial\zeta}, \tag{8}$$

where ψ is the perturbation of the flux function Ψ, the contravariant components of the perturbed magnetic field, expressed by these test functions, are given by

$$\tilde{B}^i = \frac{1}{2\pi\sqrt{g^r}}\left\{\frac{\partial\psi}{\partial\theta}, \frac{\partial\lambda}{\partial\zeta} - \frac{\partial\psi}{\partial a}, -\frac{\partial\lambda}{\partial\theta}\right\}. \tag{9}$$

Knowing that the most unstable perturbations are incompressible, $\nabla\cdot\vec{\xi} = 0$, we can write

$$\frac{\partial(\sqrt{g^r}\xi^\theta)}{\partial\theta} - \frac{\Phi'}{\Psi'}\frac{\partial(\sqrt{g^r}\xi^\theta)}{\partial\zeta} = -\frac{\partial(\sqrt{g^r}\xi^a)}{\partial a} - \frac{\sqrt{g^r}}{\Psi'}\frac{\partial\lambda}{\partial\zeta} + \frac{\sqrt{g^r}}{\Psi'}\frac{\partial^2 u}{\partial a\partial\zeta}, \tag{10}$$

and replace ξ^θ. The remaining perturbation ξ^a in the expression of W can be replaced by using relations (8) and (7).

By writing the general expression of the potential energy W in terms of the test functions instead of the displacement, one obtains

$$
\begin{aligned}
W = \frac{1}{2\mu_0}\int da d\theta d\zeta \Bigg\{ & \frac{g_{11}^r}{\sqrt{g^r}}\left(\frac{\partial\psi}{\partial\theta}\right)^2 - 2\frac{g_{12}^r}{\sqrt{g^r}}\frac{\partial\psi}{\partial\theta}\frac{\partial\psi}{\partial a} + \frac{g_{22}^r}{\sqrt{g^r}}\left(\frac{\partial\psi}{\partial a}\right)^2 + \\
& +\psi\frac{\partial u}{\partial\theta}\left[\left(\frac{F'}{\Psi'}\right)' + \left(\frac{4\pi^2 p'}{\Psi'}\right)'\frac{\sqrt{g^r}}{\Phi'}\right] - \frac{\partial}{\partial a}\left(\psi\frac{J'+\partial\nu/\partial\theta}{\Phi'}\frac{\partial u}{\partial\theta}\right) - \\
& -\frac{4\pi^2 p'}{\Psi'}\frac{\partial u}{\partial\zeta}\left[\frac{\partial u}{\partial a}\frac{\partial}{\partial\theta}\frac{g_{33}^r}{F} - \frac{\partial u}{\partial\theta}\frac{\partial}{\partial a}\frac{g_{33}^r}{F}\right] + \frac{g_{33}^r}{\sqrt{g^r}}\left(\frac{\partial\lambda}{\partial\theta}\right)^2 - \\
& +2\frac{g_{12}^r}{\sqrt{g^r}}\frac{\partial\psi}{\partial\theta}\frac{\partial\lambda}{\partial\zeta} - 2\frac{g_{22}^r}{\sqrt{g^r}}\frac{\partial\psi}{\partial a}\frac{\partial\lambda}{\partial\zeta} + \frac{g_{22}^r}{\sqrt{g^r}}\left(\frac{\partial\lambda}{\partial\zeta}\right)^2 - \\
& -2\frac{J'+\partial\nu/\partial\theta}{\Phi'}\frac{\partial u}{\partial\theta}\frac{\partial\lambda}{\partial\zeta} + 2\frac{F'}{\Phi'}\frac{\partial u}{\partial\theta}\frac{\partial\lambda}{\partial\theta}\Bigg\}.
\end{aligned} \tag{11}
$$

In the following, we develop first the values in Fourier series, perform an Euler minimisation of the energy functional and then integrate with respect to the angles θ and ζ the result of that minimisation.

The following Fourier series will be considered

$$
\begin{aligned}
&\psi = \sum_{m=-\infty}^{\infty} Y_m(a)\mathrm{e}^{\mathrm{i}(m\theta - N\zeta)}, \quad u = \sum_{m=-\infty}^{\infty} U_m(a)\mathrm{e}^{\mathrm{i}(m\theta - N\zeta)}, \\
&\lambda = \sum_{m=-\infty}^{\infty} \Lambda_m(a)\mathrm{e}^{\mathrm{i}(m\theta - N\zeta)},
\end{aligned} \tag{12}
$$

where m and N are the poloidal and the toroidal wave numbers, respectively (the "classical" notation n for the toroidal wave number will be used later to designate the normal component of the perturbed magnetic field). For simplicity, the second subscript N from the amplitudes Y_m, U_m, and Λ_m has been dropped.

In the expression of the potential energy, used in our stability code, we have avoided the strong singularity due to the pressure term in two ways: either by simply considering $p' = 0$ in the full plasma domain, or by considering a very local flattening of the pressure gradient profile (with the width of the plateau of the order of the linear resistive layer width) at the resonant surfaces, as in Ref. [15], keeping constant the safety factor profile and updating correspondingly the current density profile.

To exemplify our approach of determining the boundary conditions for the differential equations resulting from the Euler equations, we will consider a simplified case: $p' \equiv 0$ and $\lambda \equiv 0$. The last assumption seems to be "natural", because at $\lambda \equiv 0$ the longitudinal field B^ζ is not perturbed and, hence, λ gives either a contribution $\sim (p'r/B_\zeta^2)^2$ or $\sim a^2/r^2q^2$, which is often small [20]. q is the safety factor.

Taking into account relation (8), we can express U_m, U'_m and U''_m with the help of Y_m, obtaining thus a differential equation in Y_m only. The equation for the m^{th} harmonic is

$$\frac{d}{da}\left(f_{10}\frac{dY_m}{da}\right) - m^2 f_{30}Y_m - \frac{A'Y_m}{\mu - N/m} + \sum_{j=-\infty,\neq 0,\neq -m}^{\infty}\left\{\frac{d}{da}\left(f_{1j}\frac{dY_{m+j}}{da}\right) - \right.$$
$$\left. - m(m+j)f_{3j}Y_{m+j} + (m+j)\frac{d}{da}\left(f_{2j}Y_{m+j}\right) + mf_{2j}\frac{dY_{m+j}}{da}\right\} = 0, \tag{13}$$

or

$$f_{10}Y''_m + \sum_{j=-\infty,\neq 0,\neq -m}^{\infty} f_{1j}Y''_{m+j} = \left(m^2 f_{30} + \frac{A'}{\mu - n/m}\right)Y_m +$$
$$+ \sum_{j=-\infty,\neq 0,\neq -m}^{\infty}(m(m+j)f_{3j} - (m+j)f'_{2j})Y_{m+j} - f'_{10}Y'_m - \tag{14}$$
$$- \sum_{j=-\infty,\neq 0,\neq -m}^{\infty}(f'_{1j} + (2m+j)f_{2j})Y'_{m+j}.$$

$A' = j'(a)R_0/B_0$, with $j(a)$ the toroidal plasma current density, R_0 the plasma major radius and its corresponding toroidal magnetic field B_0. $\mu = -\Psi'/\Phi'$ is the rotational transform.

The f_{ij} coefficients, used in the (a, θ, ζ) coordinate system with straight field lines, can be calculated with the help of the following relations

$$f_{1j} = \left\langle \frac{g_{22}}{rD}\cos(j\theta)\right\rangle_\omega, \qquad f_{2j} = \left\langle\left(\frac{g_{12}}{r^2\langle D/r\rangle_\omega} - \frac{g_{22}\theta_a}{rD}\right)\sin(j\theta)\right\rangle_\omega,$$
$$f_{3j} = \left\langle \frac{g_{11}\theta_\omega - 2g_{12}\theta_\omega\theta_a + g_{22}\theta_a}{rD}\cos(j\theta)\right\rangle_\omega, \tag{15}$$

where the metric coefficients g_{ik}, written in a general coordinate system (a, ω, ζ), are given by an equilibrium solver [18]. With $\langle\ldots\rangle_\omega$ the conventional averaging with respect to the poloidal angle ω is implied.

In practical calculations, the number of harmonics necessary to obtain a convergence of 1% in the calculation of Δ' depends strongly on the considered problem and varies from 15 to 25.

In vector form, the system of equation (14) looks like

$$\mathbf{Y}'' = \mathbf{f}^{-1} \cdot \left(\mathbf{G} \cdot \mathbf{Y} + \mathbf{V} \cdot \mathbf{Y}' \right) + S\vec{\delta}_m (a - a_{m/N}), \tag{16}$$

where **f**, **V** and **G** are matrices, and **Y** is the flux function perturbation vector.

Note that the non-diagonal terms from the **f**, **G**, and **V** matrices represent both toroidicity and shape coupling effects.

Having practically three boundary conditions: $\mathbf{Y}(a)$, $\mathbf{Y}'(a)$ with $a \to 0$ and $\mathbf{Y}(\infty) = 0$ (or, in our case $\mathbf{Y}'(a)/\mathbf{Y}(a)|_{a=1} = \text{given}$), we can solve our system of ordinary differential equations either as a Cauchy problem or as a Two Point Boundary Value one. We have chosen the second approach and

To obtain a well conditioned system, in the r.h.s. of equations (16) a source term has been added, where S is a constant while $\vec{\delta}_m (a - a_{m/N})$ is the Dirac operator vector with a single non-vanishing element, that corresponding to the resonance surface of the m^{th} mode.

This source term S allow us to have a jump in $\mathbf{Y}'$ but with continuous **Y** across the resonance surfaces. This constant determines the amplitude of the solution **Y** and can be different for each harmonic. For a very large domain of its value, a convergence in the calculation of Δ' is ensured. In our approach we have to consider a single main harmonic at once and consider the remaining ones as satellites acting via toroidal and shaping couplings and afterwards to consider another harmonic as the main one and the remaining as satellites.

To solve the system of coupled ordinary differential equations (16), a sweep method based on a fourth-order Runge-Kutta integration scheme with an adaptive step-size has been used. Two fundamental solutions are started: one from the magnetic axis to the resonance surface, and another from the plasma boundary toward the reference singular surface.

3. BOUNDARY CONDITIONS FOR THE MODE EQUATIONS

Close to the magnetic axis ($a \to 0$), we have found the following behaviour of the amplitude of the flux function perturbation

$$Y^m(a) \sim a^m - \frac{2a^{m+2}}{(m+1)a_{m/N}^2} - \frac{(m-1)a^{m+4}}{(m+1)(m+2)a_{m/N}^4}, \tag{17}$$

where $a_{m/N}$ is the resonance radius corresponding to the wave numbers m and N respectively. Obviously, we know also the behaviour of $Y'^m(a)$ near the magnetic axis.

Due to the presence of the separatrix, the boundary conditions for the second order differential equation governing tearing mode stability, can no longer be taken, as is usual, at the magnetic axis and at infinity (practically, at three plasma-radii). Therefore, we have to consider a "natural" boundary condition just at the plasma boundary [22].

From potential theory we know that a continuous surface distribution of simple sources extending over a not necessarily closed Liapunov surface ∂D [23] and of density $\sigma(\mathbf{q})$, generates a simple-layer potential at $\mathbf{p}$, in ∂D

$$\Phi(\mathbf{p}) = \int_{\partial D} \sigma(\mathbf{q}) g(\mathbf{p},\mathbf{q}) dq,$$

where $g(\mathbf{p},\mathbf{q}) = 1/\left|\mathbf{r_p} - \mathbf{r_q}\right|$, with $\mathbf{r}$ the position vector, is the three-dimensional free space Green's function. Roughly speaking a Liapunov surface has a continuously varying tangent plane at each point, but it does not necessarily possess a curvature everywhere and is slightly less general than a Kellogg regular surface [23]. This potential is continuous everywhere, is differentiable to the second order and satisfies Laplace's equation and is therefore a harmonic function everywhere except at ∂D. If σ is Hölder continuous at $\mathbf{p} \in \partial D$, then the tangential derivatives of Φ exist and are continuous at $\mathbf{p}$, while the normal derivatives of Φ exist and are discontinuous [24–26].

If we draw a normal line to one side of ∂D at $\mathbf{p}$ and locate points on the normal by a variable n which increases moving away from ∂D, then at any point on the normal, other than $\mathbf{p} \in \partial D$, we have

$$\frac{\partial \Phi(\mathbf{p})}{\partial n} = \int_{\partial D} \frac{\partial g(\mathbf{p},\mathbf{q})}{\partial n} \sigma(\mathbf{q}) dq,$$

where $\partial g(\mathbf{p},\mathbf{q})/\partial n$ denotes the derivative of g at $\mathbf{p}$, keeping $\mathbf{q}$ fixed. At $\mathbf{p}$ we have

$$\frac{\partial \Phi(\mathbf{p})}{\partial n} = \int_{\partial D} \frac{\partial g(\mathbf{p},\mathbf{q})}{\partial n} \sigma(\mathbf{q}) dq - 2\pi\sigma(\mathbf{p}); \qquad \mathbf{p} \in \partial D.$$

With $\tilde{\mathbf{B}} = -\nabla\Phi$, and $\partial\Phi(\mathbf{p})/\partial n = -\tilde{B}_n$ we can write the basic relation of the magnetic field produced by a general surface charge distribution on a toroidal closed surface ∂D:

$$2\pi\sigma(l,\zeta) + \oint\oint \sigma(l',\zeta') b_n(l,l',\zeta'-\zeta) dl' d\zeta' = \tilde{B}_n(l,\zeta), \tag{18}$$

with l a contour coordinate, and $b_n(l,l',\zeta'-\zeta)$ the normal magnetic field given by a unit surface charge. The $(l) \equiv (r,z)$ coordinates represent the field point $\mathbf{p}$, while the $(l') \equiv (r',z')$ coordinates represent the source point $\mathbf{q}$. In our attempt to find a surface charge distribution that gives the magnetic field $\tilde{B}_n$, we have to keep in mind that as with any magnetic field, the normal components are continuous, while the tangential components present a jump (due to surface currents). In contrast, the tangential derivatives of Φ are continuous, while the normal ones jump on ∂D due to the presence of the surface charge.

Assuming the following "classical" dependencies on ζ for fields and charges in a cylindrical coordinate system (r, z, ζ)

$$\tilde{B}_n(r, z, \zeta) = \tilde{B}_{nN}(l)\mathrm{e}^{-\mathrm{i}N\zeta}, \quad \sigma(r, z, \zeta) = \sigma_N(l)\mathrm{e}^{-\mathrm{i}N\zeta}, \tag{19}$$

Eq. (18) becomes

$$2\pi\sigma_N(l) + \oint\oint \sigma_N(l')\mathrm{e}^{-\mathrm{i}Nu}\, b_n(l, l', u)dl'du = \tilde{B}_{nN}(l), \tag{20}$$

where $u = \zeta' - \zeta$. Making the notation

$$b_{nN}(l, l') = \oint \mathrm{e}^{-\mathrm{i}Nu} b_n(l, l', u)du, \tag{21}$$

we obtain finally

$$2\pi\sigma_N(l) + \oint \sigma_N(l')\, b_{nN}(l, l')dl' = \tilde{B}_{nN}(l). \tag{22}$$

The normal and tangential field components are given by

$$b_{nN}(l, l') = n_r(l)b_{rN}(l, l') + n_z(l)b_{zN}(l, l'), \tag{23}$$

and

$$b_{\tau N}(l, l') = \tau_r(l)b_{rN}(l, l') + \tau_z(l)b_{zN}(l, l'), \tag{24}$$

where n_r and n_z are the components of the normal vector, $\tau_r = -n_z$ and $\tau_z = n_r$ are the components of the tangential vector, while b_{rN} and b_{zN} are the r and z components of the magnetic field given by a unit surface charge. On the other hand, the tangential and the normal components of the perturbed magnetic field are related to the perturbed magnetic flux by the relations

$$\tilde{B}_\tau = \frac{\nabla a \times \nabla\zeta}{|\nabla a \times \nabla\zeta|} \cdot \tilde{\mathbf{B}} = \frac{\mathbf{e}_\theta}{|\mathbf{e}_\theta|} \cdot \tilde{\mathbf{B}} = -\frac{\sqrt{g_{22}^r}}{\sqrt{g^r}} \frac{\partial\psi}{\partial a} + \frac{g_{12}^r}{\sqrt{g^r g_{22}^r}} \frac{\partial\psi}{\partial\theta}, \tag{25}$$

and

$$\tilde{B}_n = \frac{\nabla a}{|\nabla a|} \cdot \tilde{\mathbf{B}} = \frac{1}{\sqrt{g_{22}^r g_{33}^r}} \frac{\partial\psi}{\partial\theta}. \tag{26}$$

In fact, these two relations hold for any coordinate system.

Let us assume that charges are distributed on the $a = 1$ surface (the plasma boundary) according to the distribution

$$\sigma_N(l(\theta)) = \sum_{m=1}^{M} C_{mN} \mathrm{e}^{\mathrm{i}m\theta}, \tag{27}$$

where, C_{mN}'s are the unknown complex amplitudes of the charges.

Using the notation

$$\tilde{B}_{nmN}(l(\theta)) = 2\pi \mathrm{e}^{\mathrm{i}m\theta(l)} + \oint \mathrm{e}^{\mathrm{i}m\theta(l')} b_{nN}(l,l')dl', \tag{28}$$

and

$$\tilde{B}_{\tau mN}(l(\theta)) = \oint \mathrm{e}^{\mathrm{i}m\theta(l')} b_{\tau N}(l,l')dl', \tag{29}$$

the resulting field components, due to all the charges, and considering Eq. (12) for ψ, are:

$$\tilde{B}_{nN} = \sum_{m=1}^{M} C_{mN} \tilde{B}_{nmN} = \frac{\mathrm{i}}{\sqrt{g_{22}^r g_{33}^r}} \sum_{m=1}^{M} mY_m \mathrm{e}^{\mathrm{i}m\theta}, \tag{30}$$

and

$$\begin{aligned} \tilde{B}_{\tau N} &= \sum_{m=1}^{M} C_{mN} \tilde{B}_{\tau mN} = -\sqrt{g_{22}^r / g^r} \sum_{m=1}^{M} Y'_m \mathrm{e}^{\mathrm{i}m\theta} + \mathrm{i} \frac{g_{12}^r}{\sqrt{g^r g_{22}^r}} \sum_{m=1}^{M} mY_m \mathrm{e}^{\mathrm{i}m\theta} = \\ &= -\sqrt{g_{22}^r / g^r} \sum_{m=1}^{M} Y'_m \mathrm{e}^{\mathrm{i}m\theta} + \frac{g_{12}^r}{\sqrt{g^r g_{22}^r}} \sqrt{g_{22}^r g_{33}^r} \sum_{m=1}^{M} C_{mN} \tilde{B}_{nmN}, \end{aligned} \tag{31}$$

or

$$\sum_{m=1}^{M} C_{mN} \left(\tilde{B}_{\tau mN} - \frac{g_{12}^r}{\sqrt{g^r g_{22}^r}} \sqrt{g_{22}^r g_{33}^r} \tilde{B}_{nmN} \right) = -\sqrt{g_{22}^r / g^r} \sum_{m=1}^{M} Y'_m \mathrm{e}^{\mathrm{i}m\theta}. \tag{32}$$

In vector form

$$\mathbf{D} \cdot \mathbf{C} = \mathbf{Y}; \qquad \mathbf{F} \cdot \mathbf{C} = \mathbf{Y}', \tag{33}$$

where $\mathbf{D}$ and $\mathbf{F}$ are $[M \times M]$ complex matrices, with the elements given by

$$d_{kj} = -\frac{\mathrm{i}}{k} \left\langle \sqrt{g_{22}^r g_{33}^r} B_{njN} \mathrm{e}^{-\mathrm{i}k\theta} \right\rangle_\theta, \tag{34}$$

$$f_{kj} = -\left\langle \sqrt{g^r / g_{22}^r} \left(B_{\tau jN} - \frac{g_{12}^r}{\sqrt{g^r g_{22}^r}} \sqrt{g_{22}^r g_{33}^r} \tilde{B}_{njN} \right) \mathrm{e}^{-\mathrm{i}k\theta} \right\rangle_\theta, \tag{35}$$

$k, j = 1, \ldots, M$, $\mathbf{C} = [C_{1N}, C_{2N}, \ldots, C_{MN}]^T$ and $\mathbf{Y} = [Y_1, Y_2, \ldots, Y_M]^T$ are $[M]$ vectors.

By eliminating the $\mathbf{C}$ vector, one obtains

$$\mathbf{Y}' = \mathbf{F} \cdot \mathbf{D}^{-1} \cdot \mathbf{Y}. \tag{36}$$

For the numerical integration of Eq. (16) a sweep method has been used

$$\mathbf{Y}_k = \vec{\alpha}_k \cdot \mathbf{Y}_{k+1} + \vec{\beta}_k, \tag{37}$$

with $\vec{\alpha}_k$ a known $[M \times M]$ coefficient matrix and $\vec{\beta}_k$ a known $[M]$ coefficient vector, both resulting from a forth-order Runge-Kutta integration scheme.

Thus, the boundary condition at the plasma boundary becomes

$$\mathbf{Y}_{k+1} = \left(\mathbf{I} - h\mathbf{F} \cdot \mathbf{D}^{-1} - \vec{\alpha}_k\right)^{-1} \cdot \vec{\beta}_k, \tag{38}$$

with **I** the unit matrix and h the "radial" integration mesh. Note that the boundary conditions for **Y** are the result of a poloidal coupling of all surface charge distributions C_{mN}.

To calculate the $\Delta'_{m/N}$ stability parameter, the classical definition has been used

$$\Delta'_{m/N} = \Re\left(\left.\frac{Y'_m}{Y_m}\right|_{a_{m/N}+\epsilon} - \left.\frac{Y'_m}{Y_m}\right|_{a_{m/N}-\epsilon} \right). \tag{39}$$

For unit perturbations $Y_{2/1}$ $(m = 2,\ N = 1)$ and $Y_{3/2}$ $(m = 3,\ N = 2)$, the corresponding surface charge distributions $C_{2/1}$ and $C_{3/2}$ are given in Fig. 1.

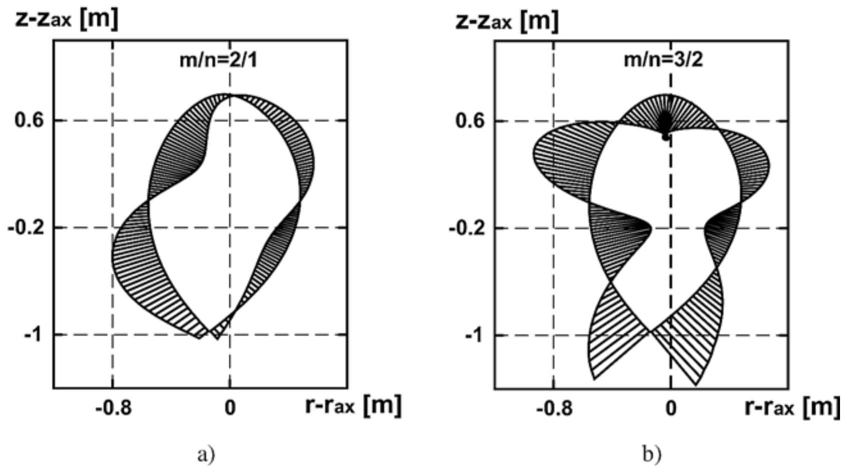

Fig. 1 – The surface charge distribution along the plasma boundary for unit flux perturbations $Y_{2/1}$ and $Y_{3/2}$, respectively. The plasma configuration of the ASDEX Upgrade tokamak corresponding to the shot no. 13476 at 5.2 s has been considered.

4. CONCLUSIONS

In this paper the calculation of tearing modes in toroidally diverted plasma configurations are given, with particular attention paid to the boundary conditions. These results are based mainly on our previous paper [16].

Due to the separatrix, the determination of the flux function perturbation (necessary for stability calculations) can no longer make use of the boundary conditions of this perturbation at infinity. Starting from potential theory, a "natural" boundary condition for the flux function perturbation, just at the plasma boundary, has been deduced. This approach of fixing "natural" boundary conditions on the separatrix can be used also for tearing modes calculation in the Rutherford regime and at their nonlinear saturated stage.

As an example of application of our approaches, a particular equilibrium configuration of the ASDEX Upgrade tokamak has been considered and a detailed investigation of the dependence of the tearing stability parameter Δ' on plasma shape is given for a realistic tokamak equilibrium. The results shown above are at least in qualitative agreement with experimental observations on ASDEX Upgrade [27] and JET [28] of a stabilizing influence of triangularity.

The knowledge of Δ' for realistic tokamak plasmas is especially important for an understanding of the plasma stability against NTMs [29].

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