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BOUNDARY CONDITION METHOD ACCORDING TO REDUCED SCATTERING LENGTH

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Abstract. The reactions systems are studied according to the Boundary Conditions Method. The scattering of a low-energy particle on two scatterers is described by a system of two coupled equations; the corresponding single channel interactions are described by one-channel scattering lengths. The problem of two-channel scattering could be described also in terms of one-channel scattering by means of Reduced Scattering Length. The Reduced Scattering Length takes into account the effect of eliminated channel *via* its scattering length as well as by multiple scattering of incident particle on two scatterers. One demonstrates that a virtual two-body state becomes bound in a three-body system, provided all scattering lengths are negative.

Key words: boundary condition, scattering length, three-body systems, low-energy.

1. INTRODUCTION

The study of multichannel and/or multilevel problems in terms of reduced or effective operators is frequently used both in Reactions and Spectroscopy fields of Atomic and Nuclear Physics. We mention here, as example, the method of Coupled Channels for Atomic Collisions or Nuclear Reactions. The system of Coupled Equations is truncated (*e.g.* by computational reasons); to compensate the eliminated channels, the interaction potential is modified. The additional terms, originating in eliminated channels, result into Effective Interaction. The formal basis for Effective Interaction is the Projector Method of Feshbach, used both in Atomic and Nuclear Physics. Two projection operators are used to divide the set of scattering channels/states into two subsets, retained and eliminated channels/states. One obtains an Effective Hamiltonian for retained channels; the "bare" interaction in the complete reaction system is replaced by an effective one acting in the retained channels space.

The method of channel elimination was firstly developed by Wigner in $R-(K-)$ Matrix Theory. The Reduced $R-(K-)$ Matrix has, in addition to terms describing retained channels, the contribution from eliminated channels too. The concept of Reduced or Effective Operator was further extended to the Scattering (Collision) Matrix, because the $S-(U-)$ Matrix is primary object in Scattering Physics.

In the present work we extended the Effective Operators to Reduced Scattering Length. The concept of Scattering Length was introduced by Fermi in order to describe phenomena developing in the bound-unbound transition zone. It describes the behaviour of the wave function near zero-energy, below, at or above breaking threshold. Moreover the Scattering Length is related to Scattering Amplitude; it defines the Scattering Amplitude just at threshold energy. The Scattering Amplitude and Scattering Potential are related, in Born Approximation, *via* Fourier Transform. In this case the corresponding potential is known as Fermi Pseudopotential; it is a Contact (Zero-Range)Potential and its magnitude is proportional to Scattering Length. This way, a relation of Contact Potential and Scattering Length to Boundary Condition Method was established. The Contact Interaction is described in Quantum Mechanics through the Logarithmic Derivative of the wave function (Boundary Condition) in origin. The basic ideea of the Contact Potential Method is to replace the interaction of the particles by corresponding Boundary Conditions [1].

This description was generalized to many-channel case. A physical example, in this respect is scattering of a particle on two scatterers. Recently we applied [2] the Boundary Condition Method for describing Borromean Nuclei: loosely three-body systems consisting of a core and two neutrons which become unbounded when one neutron is removed. In that work we studied how a single channel virtual state becomes a bound one in a two-channels reaction system.

In the present work we study formally the problem for two reaction channels. We reduce the two-reaction channels problem to one-channel problem by taking into account the effect of eliminated channel in terms of Reduced Scattering Length. The Reduced Scattering Length consists from retained one-channel Scattering Length modulated by a term depending on the eliminated channel Scattering Length as well as on Multiple Scatterings between the two channels.

A physical problem related to the present work is description of the bound states of two interacting particles which are close to zero energy in the presence of a core. For such loosely bound states one can take into account the presence of the core by its scattering properties at zero energy, *i.e.* the particle-core scattering length. In the same way the particle-particle interaction is described by the particle-particle scattering length. In this paper the term *Borromean* is used to represent a state of a three particle loosely bound system which becomes unbound if one particle is removed. The formal results will be illustrated by taking the *Borromean* system as a prototype.

2. CONTACT INTERACTION, BOUNDARY CONDITION METHOD AND SCATTERING LENGTH

Let us consider a low energy level $E = -|E_0|$ in a potential well *U*, obeying condition $E_0/U \to 0$. It is well known, *e.g.* [3], that the level's position in a potential well of magnitude *U* and radius *R*, depends on the product UR^2. In this respect the level's energy is not changed in the limit $U \to \infty$ and $R \to 0$, maintaining the condition $UR^2 = constant$. This way one can study the level's properties by means of contact singular potentials ($U \to \infty$, $R \to 0$). Such singular contact potential is described by a point-like interaction, $B\delta(\vec{r})$, (Zero Range Potential Model). In the following we will relate the study of Contact Potential to Boundary Conditions Method; this method was developed mainly in Soviet Literature, *e.g.* [3–7].

Consider the matching of the wave function inside potential well, $\chi_1 = (r\Psi)_1$ to that of outside potential, $\chi_2 = (r\Psi)_2$, $\chi_2 \sim exp(-\alpha r)$, at the radius *R*, $(d\ln\chi/dr)|_R = -\alpha$. As the level's energy does not change in the above limit, $U \to \infty$, $R \to 0$, this procedure should not result into a modification of the above boundary condition. The states in the deep potential well could be described in terms of logarithmic derivative $d\ln\chi/dr$, calculated at origine, $R \to 0$. The description in terms of the logarithmic derivative appears to be a general property of Schroedinger equation.

The basic ideea of Contact or Zero Range Potential Models is to replace the solution of the Schroedinger equation inside the potential well by Boundary Conditions on the Wave Function at centre of the potential well

$$d/dr(r\Psi)|_{R=0} = -\alpha(r\Psi)|_{R=0} . \tag{1}$$

For a bound state $\chi = r\Psi \sim exp(-\alpha r)$, $\alpha = \sqrt{-2mE/\hbar^2}$, one obtains $E = -\hbar^2\alpha^2/2m$. For a scattering state with energy $E = \hbar^2k^2/2m$, the wave function is given by asymptotic boundary condition,

$$\Psi = exp(i\vec{k}\vec{r}) + f\exp(ikr)/r. \tag{2}$$

The boundary condition does relate the scattering amplitude, *f*, to the α parameter,

$$f = -1/(\alpha + ik). \tag{3}$$

One obtains that the negative of zero-energy scattering amplitude is related to logarithmic derivative

$$-f(0) = 1/\alpha = a. \tag{4}$$

The quantity $a = -1/(d\ln\chi/dr)|_{R=0}$ is called Scattering Length; for a bound state it is positive, $a > 0$, while for a virtual state it is negative, $a < 0$. The boundary condition for both cases is $d\chi/dr\,|_0 = -1/a\chi\,|_0$.

The scattering amplitude, in Born Approximation, is the Fourier Transform of the interaction potential. For a contact potential, $V(r) = B\delta(\vec{r})$, the scattering amplitude becomes

$$f = -a = -mB/2\pi\hbar^2 \tag{5}$$

resulting in a relation connecting the potential strength and scattering length, $B \sim a$. The Schroedinger equation's Zero-Range Potential becomes now, $U(\vec{r}) = 4\pi a\delta^3(\vec{r})$ and it is known as Fermi's Pseudopotential, [8]. The Fermi Pseudopotential is repulsive for a nuclear medium described by a positive scattering length. The Fermi Pseudopotential is attractive for a nuclear medium with a negative scattering length. In concrete terms a slow neutron is attracted by a virtual state, ($a < 0$), and it is repelled by a bound state, ($a > 0$). These physical results were used in confining ultra-cold neutrons, *e.g.* [8, 9]. For an ultracold energy less than Fermi's Pseudopotential strength the neutrons are totally reflected by material walls, provided neutron-material's nucleus interaction is given by a positive scattering length.

This idea was extrapolated to the case of a neutron outside a nuclear medium with a negative scattering length, [2]. A neutron is attracted by a nucleus provided their (interaction) scattering length is negative, *i.e.* the neutron and nucleus (core) do form a virtual state. However this (virtual) system's state is unstable and could manifest itself only via a scattering process. Now consider that another neutron does approach this (virtual state) system formed by the core-nucleus and initial neutron. The additional neutron and core-nucleus is another virtual-state system. Also the two-neutrons system is a virtual state one because *n-n* scattering length is negative. We approach the problem of Particle "Scattering" on two "Centres" (either neutron "scatteringö on system of core plus other neutron, or the core-nucleus "scattering" on two neutrons system), in terms of the Fermi Zero Range Potential Model, *i.e.* in terms of Boundary Conditions Method.

The Boundary Conditions Method was generalized, [10], to many centres scattering problems; one introduces boundary conditions corresponding to every non-overlaping of centres. The Boundary Conditions Method describing the scattering of a particle on two scatterers (1) and (2), are

$$d/d\rho_1(\rho_1\Psi)\,|_{\rho_1=0} = -(1/a_1)(\rho_1\Psi)\,|_{\rho_1=0}, \tag{6}$$

$$d/d\rho_2(\rho_2\Psi)\,|_{\rho_2=0} = -(1/a_2)(\rho_2\Psi)\,|_{\rho_2=0}, \tag{7}$$

where $\rho_i = r - r_i$ ($i = 1, 2$). In absence of second scatterer (well) the wave function outside the first scatterer well is $\Psi_1 = C_1\exp(-\alpha\rho_1)/\rho_1$, where α it would have significance of $1/a_1$. In a similar way, outside the second well and in absence of the first one, the wave function would be $\Psi_2 = C_2 \exp(-\alpha\rho_2)/\rho_2$, now with $\alpha \sim 1/a_2$ significance. If the both centres are present, then

$$\Psi = C_1 \exp(-\alpha\rho_1)/\rho_1 + C_2 \exp(-\alpha\rho_2)/\rho_2, \tag{8}$$

where α has to be determined by procedures of the Boundary Conditions Method. By using this wave function Ψ in above Boundary Conditions Equations, one obtains a set of two equations for the constants C_1 and C_2,

$$(\alpha - 1/a_1)\cdot C_1 + \exp(-\alpha R)/R \cdot C_2 = 0, \tag{9}$$

$$\exp(-\alpha R)/R \cdot C_1 + (\alpha - 1/a_2)\cdot C_2 = 0, \tag{10}$$

with $R = |\vec{r}_1 - \vec{r}_2|$ as a distance parameter for the two scattering centres. The solution of this set of two equations implies the determinant condition

$$(\alpha - 1/a_1)(\alpha - 1/a_2) - \exp(-2\alpha R)/R^2 = 0. \tag{11}$$

The energies of the particle in the fields of two centres is $E_i = -\hbar^2\alpha_i^2(R)/2m$, where α_1 and α_2 are roots of the above equation and m is particle (reduced) mass.

In the scattering case the trial wave function consists of two waves with amplitudes $A_1(\vec{k})$ and $A_2(\vec{k})$ radiated from the two scattering centers

$$\Psi_{sc} = A_1(\vec{k})\exp(ik\rho_1)/\rho_1 + A_2(\vec{k})\exp(ik\rho_2)/\rho_2. \tag{12}$$

The full scattering wave function contains, in addition to Ψ_{sc}, the incoming plane wave $\exp(i\vec{k}\vec{r})$. From the boundary conditions one similarly obtains equations for the effective scattering amplitudes A_1 and A_2,

$$A_1(ik + 1/a_1) + A_2 \exp(ikR)/R = -\exp(i\vec{k}\vec{R}/2). \tag{13}$$

$$A_1 \exp(ikR)/R + A_2(ik + 1/a_2) = -\exp(-i\vec{k}\vec{R}/2), \tag{14}$$

the two centers being located at $\vec{r}_1 = +\vec{R}/2$ and $\vec{r}_2 = -\vec{R}/2$, respectively. Observe that by setting $k \to i\kappa$ one obtains the bound state problem except for the right-hand terms, originating from the wave radiation of the two scatterers.

We note that in the above system of equations the matrix

$$\begin{pmatrix} ik + 1/a_1 & \exp(ikR)/R \\ \exp(ikR)/R & ik + 1/a_2 \end{pmatrix} \tag{15}$$

plays the role of the Jost Matrix for a two-reaction-channels system, [6]. Its determinant

$$(ik+1/a_1)\cdot(ik+1/a_2)-\exp(2ikR)/R^2=0 \tag{16}$$

defines the poles k which correspond to quasistationary states of the system. One can put into correspondence this formal result to the following physical two-channels reaction system: The target system has two constituents on which a particle is scattered. The two constituents are interpreted as two "states" of the target system. The particle is elastically scattered on both components of the target. Setting the Jost matrix, eq. (15), proportional to $ik+M^{-1}$, the interaction with the two-states target is described by a "Scattering Length Matrix" M^{-1}

$$M^{-1}=\begin{pmatrix} 1/a_1 & \exp(ikR)/R \\ \exp(ikR)/R & 1/a_2 \end{pmatrix}. \tag{17}$$

Indeed, by comparison with eq. (4), it is the generalization of the one-channel scattering length. The Scattering Length Matrix has the following explicit form

$$M=\left[1-\frac{a_1}{R}\cdot\frac{a_2}{R}\cdot\exp(2ikR)\right]^{-1}\begin{pmatrix} a_1 & -a_1\cdot a_2\dfrac{\exp(ikR)}{R} \\ -a_2\cdot a_1\dfrac{\exp(ikR)}{R} & a_2 \end{pmatrix}. \tag{18}$$

It is seen that the non-diagonal coupling term, $a_1\cdot a_2\dfrac{\exp(ikR)}{R}$, between the two target-states is proportional to the two scattering lengths and decreases with intercenter distance *via* the radiation condition. The scattering length matrix thus includes the effects of multiple scatterings of the wave between the two scatterers. In particular, the term in square brackets, $\left[1-\frac{a_1}{R}\cdot\frac{a_2}{R}\cdot\exp(2ikR)\right]$, is due to multiple scattering corrections, [11]. It becomes more important as the scattering lengths exceed the intercenter spacing.

We remark that in the above interpretation the coupling terms are not directly related to the scattering length of the two constituents of the target and thus to the interaction between them. However, the two scatterers are not free and they interact *via* multiple scattering of the spherical wave of one center reaching the other one. Thus the rescattering coupling terms effectively include the mutual scattering length of the two constituents of the target.

3. REDUCED SCATTERING LENGTH

Scattering Amplitude. The Scattering Length Matrix M^{-1} is defined by

$$M^{-1}+ik=\begin{pmatrix} ik+1/a_1 & \exp(ikR)/R \\ \exp(ikR)/R & ik+1/a_2 \end{pmatrix}. \tag{19}$$

(Compare to One Channel case, $A = -(1/a + ik)^{-1}$).

The Scattering Amplitude Matrix for two channels is

$$A = -(M^{-1} + ik)^{-1} \tag{20}$$

The Reduced Scattering Length $\tilde{a}_1$ for particle-scatterer is obtained by formal elimination of other scatterer:

$$1/\tilde{a}_1 = 1/a_1 - \frac{\exp(ikR)}{R} \frac{1}{1/a_2 + ik} \frac{\exp(ikR)}{R} \tag{21}$$

$$\tilde{a}_1 = \frac{a_1}{1 - \frac{a_1}{R} \cdot \frac{a_2}{R} \cdot \frac{\exp(2ikR)}{1 + ika_2}}. \tag{22}$$

The Scattering Amplitude Matrix for two equations (two channels),

$$A = -(M^{-1} + ik)^{-1} \tag{23}$$

$$A_{11} = -(ik + 1/\tilde{a}_1)^{-1}$$

$$A_{12} = (ik + 1/\tilde{a}_1)^{-1} \frac{\exp(ikR)}{R} (ik + 1/a_2)^{-1}$$

corresponding to the two Scattering Equations, (13) and (14), results into the Effective Scattering Amplitude(s) A_1 (and A_2)

$$A_1 = A_{11} \exp(i\vec{k}\vec{R}/2) + A_{12} \exp(-i\vec{k}\vec{R}/2), \tag{24}$$

$$A_1 = -(ik + 1/\tilde{a}_1)^{-1} \left[\exp(i\vec{k}\vec{R}/2) - \frac{\exp(ikR)}{R} (ik + 1/a_2)^{-1} \exp(-i\vec{k}\vec{R}/2) \right]. \tag{25}$$

The physical meaning of the terms in Effective Scattering Amplitude is, (for multiple scattering, see also [11]):

$\exp(i\vec{k}\vec{R}/2)$ – incident plane wave on center 1

$\exp(-i\vec{k}\vec{R}/2)$ – incident plane wave on center 2

$-(ik + 1/a_2)^{-1}$ – scattering amplitude for center 2

$\frac{\exp(ikR)}{R}$ – spherical wave emitted by center 2 (or 1) and impinging center 1 (or 2)

$\left[\exp(i\vec{k}\vec{R}/2) - \frac{\exp(ikR)}{R} (ik + 1/a_2)^{-1} \exp(-i\vec{k}\vec{R}/2) \right]$ – effective wave impinging center 1

$-(ik + 1/\tilde{a}_1)^{-1}$ – effective scattering amplitude for center 1.

Reduced Scattering Length. The coupling or rescattering effect can also bee seen at the level of one component of the scattering amplitude matrix in terms of "Reduced Scattering Length". This is the particle scattering length with respect to the first scatterer by taking into account the effect of the second one,

$$1/a_1 \to 1/\tilde{a}_1 = 1/a_1 - \frac{\exp(ikR)}{R}\frac{1}{1/a_2 + ik}\frac{\exp(ikR)}{R}, \tag{26}$$

$$a_1 \to \tilde{a}_1 = \frac{a_1}{1 - \frac{a_1}{R}\cdot\frac{a_2}{R}\cdot\frac{\exp(2ikR)}{1 + ika_2}}. \tag{27}$$

The reduced scattering length for the bound state case, $(k \to i\kappa)$, is then

$$\tilde{a}_1 = \frac{a_1}{1 - \frac{a_1}{R}\cdot\frac{a_2}{R}\cdot\frac{\exp(-2\kappa R)}{1 - \kappa a_2}}. \tag{28}$$

One can check, from this formula and by using eq. (11) for bound state case, that the inverse of the $\alpha(\kappa)$ parameter has the meaning of the reduced or effective scattering length.

In the following we discuss the various cases for the dependence of reduced scattering length $\tilde{a}_1$ on the intercenter distance R and the scattering lengths of the two channels. Here we take the limit $\kappa = 0$, but the same qualitative results are obtained for non-zero $\kappa < 1/|a_2|$.

The Reduced Scattering Length $\tilde{a}_1$ is negative provided the scattering lengths of the two channels a_1 and a_2 are negative and also the condition $R^2 > |a_1|\cdot|a_2|$ is fulfilled. The virtual state scattering length of the first channel is thus strengthened; the reduced scattering length varies from $-|a_1|$ to $-\infty$. For the transition condition $R^2 = |a_1||a_2|$ the reduced scattering length shifts to $+\infty$; the situation of a bound state is realized. The Reduced Scattering Length of the first channel $\tilde{a}_1$ remains attractive $(a_1 < 0)$ even for repulsive scattering length of the second channel, $a_2 = +|a_2|$, but now it is limited to the interval $[-|a_1|, -0]$. The virtual state is never transformed into a bound one. For positive a_1 but negative a_2 the Reduced Scattering Length $\tilde{a}_1$ is moderately repulsive; it decreases from $+|a_1|$ to +0. The repulsive property of Reduced Scattering Length is strongly amplified, if the scattering lengths of the two channels are both repulsive. It increases from $|a_1|$ to $+\infty$ provided R^2 tends toward $|a_1|\cdot|a_2|$.

Effective Scattering Length. The scattering wave function consists of two waves with amplitudes $A_1(\vec{k})$ and $A_2(\vec{k})$ radiated from the two scattering centers

$$\Psi_{sc} = A_1(\vec{k})\exp(ik\rho_1)/\rho_1 + A_2(\vec{k})\exp(ik\rho_2)/\rho_2. \tag{29}$$

The scattered wave function in assymptotic region ($\rho_1 = \rho_2 = \rho$) becomes $A_{eff}\exp(ik\rho)/\rho$ with, $A_{eff} = A_1 + A_2$,

$$A_1 = -(ik + 1/\tilde{a}_1)^{-1}\left[\exp(i\vec{k}\vec{r}_1) - \frac{\exp(ikR)}{R}(ik + 1/a_2)^{-1}\exp(i\vec{k}\vec{r}_2)\right], \tag{30}$$

$$A_2 = -(ik + 1/\tilde{a}_2)^{-1}\left[\exp(i\vec{k}\vec{r}_2) - \frac{\exp(ikR)}{R}(ik + 1/a_1)^{-1}\exp(i\vec{k}\vec{r}_1)\right], \tag{31}$$

with

$$\tilde{a}_1 = \frac{a_1}{1 - \frac{a_1}{R}\cdot\frac{a_2}{R}\cdot\frac{\exp(2ikR)}{1 + ika_2}}, \tag{32}$$

and

$$\tilde{a}_2 = \frac{a_2}{1 - \frac{a_1}{R}\cdot\frac{a_2}{R}\cdot\frac{\exp(2ikR)}{1 + ika_1}}. \tag{33}$$

In zero-energy limit, ($k = 0$), $A_{eff}(0) = A_1(0) + A_2(0) = -a_{eff}$,

$$a_{eff} = \frac{a_1 + a_2 - 2a_1\cdot a_2/R}{1 - \frac{a_1}{R}\cdot\frac{a_2}{R}}. \tag{34}$$

The scattered wave function consists of two waves with amplitudes $A_1(\vec{k})$ and $A_2(\vec{k})$ radiated from the two scattering centers, (eq. 12). The scattered wave function in assymptotic region ($\rho_1 = \rho_2 = \rho$) becomes (with $A_{eff} = A_1 + A_2$)

$$\Psi_{sc} = A_{eff}\exp(ik\rho)/\rho. \tag{35}$$

In zero-energy limit, ($k = 0$), $A_{eff}(0) = A_1(0) + A_2(0) = -a_{eff}$, one obtains the Effective Scattering Length

$$a_{eff} = \frac{a_1 + a_2 - 2a_1\cdot a_2/R}{1 - \frac{a_1}{R}\cdot\frac{a_2}{R}}. \tag{36}$$

The Effective Scattering Length is sum of the two reduced scattering lengths modulated by the magnitudes of impinging effective waves.

The Effective Scattering Length describes how the neutron n is bound by the two body system $n - C$. The Effective Scattering Length a_{eff} is negative provided the scattering lengths of the two channels a_1 and a_2 are negative and also the condition $R^2 > |a_1| \cdot |a_2|$ is fulfilled. The virtual state is thus strengthened; the effective scattering length varies from R to $-|a_1| - |a_2|$ to $-\infty$. For the transition condition $R^2 = |a_1||a_2|$ the Effective Scattering Length shifts to $+\infty$; the situation of a bound state is realized.

On Reduced and Effective Scattering Lengths. The Reduced Scattering Lengths of the two centers and the Effective Scattering Length are related. The Reduced Scattering Length $\tilde{a}_1 = \tilde{a}_{nC}$ proves how the incoming neutron n is bound by the Core C (the virtual state $n - C$ becomes a bound state) due to the second neutron $\tilde{n}$. Symmetrically the $\tilde{a}_2 = \tilde{a}_{n\tilde{n}}$ Reduced Scattering Length proves the binding of the neutron n with the other neutron $\tilde{n}$. The Effective Scattering Length tells us how the neutron n is bound by the two body system $n - C$. The Effective Scattering Length involves additional constraint on numerator, *i.e.* sign of $a_1 + a_2 - 2a_1 \cdot a_2/R^2$. Perhaps this makes difference in physics, otherwise the two definitions of the Reduced and Effective Lengths seem equivalent.

Let try to compare, in zero-energy limit, the Reduced Scattering Length and Effective Scattering Length for two virtual states $a_1 = a_2 = -a$ (a positive). One obtains

$$\tilde{a}_1 = \tilde{a}_2 = \frac{-a}{1 - \frac{a}{R} \cdot \frac{a}{R}}. \tag{37}$$

and for a_{eff}

$$\tilde{a}_{eff} == \frac{-2a}{1 - \frac{a}{R}}. \tag{38}$$

Both results in same physical conclusion *i.e.* the transition from virtual state to bound state for $R = a$.

Now let compare for the case of one virtual state $a_1 = -a$ and a bound state $a_2 = a$, (a positive). One obtains

$$\tilde{a}_1 = -\tilde{a}_2 = \frac{-a}{1 + \frac{a}{R} \cdot \frac{a}{R}}. \tag{39}$$

and

$$\tilde{a}_{eff} = \frac{2a^2/R}{1 + \frac{a}{R} \cdot \frac{a}{R}}. \tag{40}$$

According to Reduced Scattering Length the center 1 still attracts but $\tilde{a}_1$ is limited between –*a* and 0. The center 2 still repels but repulsion is diminished from *a* to 0. It looks plausible from physical point of view. The Effective Scattering Length is totally repulsive. Perhaps the difference between the two quantities consist in the Effective Scattering Length is defined only in assymptotic region, and, for non-zero energy $k \neq 0$, it includes terms like $exp(i\vec{k}\vec{r}_1)$ and $exp(i\vec{k}\vec{r}_2)$, which depend on origin of coordinates.

How a Virtual State becomes a Bound State. The Reduced Scattering Length is the zero-energy scattering amplitude for the first scatterer but taking into account the influence of the second one *via* multiple scattering terms, eq. (22),

$$A_{11}(0) = -\tilde{a}_1(0) = -\frac{a_1}{1 - \frac{a_1}{R} \cdot \frac{a_2}{R}}. \tag{41}$$

We also discuss, from this equation , the effect of multiple scattering in terms of the effective Fermi Pseudopotential which is proportional to the zero-energy scattering amplitude, (*i.e.*) Reduced Scattering Length, $U_{11} \sim -A_{11}(0) \sim \tilde{a}_1$. If the two scattering lengths of the neutron with respect to the constituents of the target have different signs, $a_1 \cdot a_2 < 0$, then the denominator of this formula is always larger than unity and the effective interaction is diminished. The second scatterer thus reduces the attraction or repulsion of the particle. If the two scattering lengths have the same sign, $a_1 \cdot a_2 > 0$, the denominator could be either positive or negative, depending on the intercenter distance *R*. If the intercenter distance is large, $R^2 > a_1 \cdot a_2$, the denominator is positive and less than unity; the second scatterer will reinforce the effect of the first one and this effect becomes stronger as *R* decreases. If the two scatterers are very near, $R^2 < a_1 \cdot a_2$, the denominator has changed sign, and the attraction is replaced by repulsion or vice versa. At the transition point, $a_1 \cdot a_2/R^2 \to 1$, the Reduced (Effective) Scattering Length changes from $-\infty$to $+\infty$; the virtual state transforms into a bound state. This point defines the largest spatial dimension of the system, $R = \sqrt{a_1 a_2}$.

A potential well which just has bound a neutron will repel another neutron. This is the physical meaning of Fermi Pseudopotential; it is attractive if no bound neutron exists in the potential well and it becomes repulsive if a neutron was just bound, see Ref. 8. We can extend this reasoning to adding another neutron to a Borromean nucleus, *i.e.* to a four-body system. It follows that a Borromean nucleus cannot bind another neutron.

In order to see how the two scatterers cooperate in binding the third particle, we discuss the reduced scattering length $\tilde{a}_1$ for different bound state energies κ, eq.

(28). It depends on energy through the coupling of the second channel. The rescattering effects are dependent on energy by the factor $\sim \exp(-2\kappa \cdot R)/$ $/(1-\kappa \cdot a_2)$. For attractive scattering length of the second channel $a_2 < 0$ the energy-dependent factor $\exp(-2\kappa \cdot R)/(1+\kappa \cdot |a_2|)$ acts as damping mechanism for rescattering. Thus the damping mechanism becomes stronger as the state is more bound. The absolute magnitude of the Reduced Scattering Length $|\tilde{a}_1|$, as function on energy, decreases for attraction from $|a_1|/\left[1-\frac{|a_1|}{R}\cdot\frac{|a_2|}{R}\right]$ to the single channel limit $|a_1|$. The very deep bound states are not affected by rescattering *i.e.* the rescattering is no more effective in producing deep bound states. For the case of repulsion of the second scatterer $a_2 > 0$ and attraction of the first one $a_1 < 0$, by increasing κ from 0 to $1/a_2$, one obtains a decrease of the absolute magnitude of the reduced scattering length $|\tilde{a}_1|$ from $|a_1|$ to zero. The particle cannot become deeply bound even it is strongly attracted by first scatterer, being repelled by the second one.

This type of "coupling", (multiple scattering of wave between the two centres), is vital in the description of Borromean system. In the zero-energy limit it behaves as an "effective interaction" proportional to $-1/R^2$. It is proved, [2], that such effective interaction is necessary for the description of borromean nuclei but it is not sufficient; it should be supported by the existence of a virtual-state of the neutron with respect to the two components of the target.

Alternative Approach:Two-Channel Scattering on One Center. Another approach to Physics of Scattering Length Matrix for multichannel reactions could be a paralel analysis to Initial/Final State Interaction in terms of input/output single channel Jost functions, [12]. The inverse of single channel scattering length ($-1/a$) is analog of single channel Jost function J; both define bound and quasistationary states in a given reaction channel. The Scattering Matrix element S_{12} describing Initial and Final State Intercations for a transition between two reaction channels is proportional to both inverses of Jost functions $S_{12} \sim 1/(J_1 J_2)$. A similar formal situation one can observe in the Scattering Length Matrix related to single channel scattering lengths $M_{12} \sim a_1 a_2$.

According to above analogy one can develop an alternative description to Borromean system in terms of Boundary Condition Method. The description is not more related to single particle scattering on two centres but rather of "Two-Channel Scattering on One Centre". The scattering is described by a matrix equation

$$d/dr(r\Psi)|_{r=0} = -(M^{-1})(r\Psi)|_{r=0}, \tag{42}$$

with Ψ describing the two reaction (states) channels for colliding the two particle (neutron) system on the scattering centre (core). The interaction of each multichannel component (neutron) with scattering centre (core) is described by corresponding scattering lengths a_1 and a_2. The interaction between the two particles (neutrons) is accounted by non-diagonal terms ~ a_1a_2 of the Scattering Length Matrix. One obtains the matricial equations but with another physical interpretation: the neutrons virtual two-state system collide the core. All interactions in the three body system are attractive. However one should remark that the interaction between the two neutrons is represented not by neutron-neutron scattering length but rather by the multiple scattering terms. The multicomponent wave function Boundary Condition Method was developed by Dalitz [13], see also [5].

4. CONCLUSIONS

The Reduced Scattering Length approach was applied for studying Borromean Nuclei, *i.e.* one core and two neutrons losely bound systems, which become unbound if one neutron is removed. The Neutron-Core as well as Neutron-Neutron Scattering Lengths are negative; the corresponding two-particle states are virtual, *i.e.* the two Scattering Lengths are negative.

The results obtained allow us to conclude that one can use the Fermi Contact Pseudopotentials (Boundary Conditions Method) to describe the weakly bound states in a three-body system. This method relates the properties of the two-body interactions to the properties of the three-body bound states: spatial extension and energy. This approach to Borromean Nuclei was inspired from Ultracold Neutron Physics [8, 9]. The Fermi Pseudopotential is proportional to neutron- nuclear medium scattering length. It is repulsive provided the scattering length is positive; this results into total reflection of ultracold neutron by the material walls. We extrapolated, [2], this idea to the case of neutron interacting with nuclear medium components (a Core nucleus and another neutron) both described by negative scattering lengths. The Fermi Pseudopotential is thus negative and the neutron is attracted by the two components, provided the other neutron and Core form a virtual state (negative scattering length). This virtual system neutron-Core is however unstable. On the other hand the neutron-neutron scattering length is negative too; the “di-neutron” is unstable in free space. This three-body system appears as unbound. But the second neutron will reinforce the attraction in the two-body virtual state (Core-neutron) system, depending on its proximity parameter with respect to Core. This process is described in terms of Reduced Scattering Length. The Reduced Scattering Length neutron-Core takes into account the effect of the second neutron. It proves how a virtual state becomes a bound state, as function of the two scattering lengths (neutron-Core and neutron-neutron) and on the neutron-Core distance. The square of the critical distance, resposible for transition from virtual to bound regime, is just the product of the two scattering lengts $R^2 = a_1a_2$.

The main physical result of this approach is the demonstration how a neutron virtual state becomes a bound state in the field of two attractive scatterers; or how the neutron becomes bound in a Borromean Nucleus in spite the separate neutrons are unbound with respect to the Core. A weak point of this approach is that the interaction of the two scatterers is not explicitelly taken into account.It is simulated in terms of multiple scattering of the incident neutron on two scatterers. The problems of direct interaction of the two scatterers and of multiple wave scattering needs further investigation.

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