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SCALE RELATIVITY THEORY FOR AN ARBITRARY FRACTAL DIMENSION

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Abstract. Considering the fractal structure of space-time, a Burgers – Korteweg – de Vries (BKdV) type equation is obtained. Particularly, if the motions of the "non-differentiable fluid" are irrotational, the BKdV type equation is reduced to a non-linear Schrödinger type equation. In this case, the scalar complex velocity field simultaneously becomes wave function.

Key words: fractal space-time, non-linear type equation.

1. INTRODUCTION

The idea that the quantum space-time of microphysics is fractal, rather than flat and Minkowskian as assumed up to now, was suggested in [1, 2]. This proposal was itself based on earlier results [3–6], obtained at first by Feynman (see in particular [7] and references therein), concerning the geometrical structure of quantum paths. These studies have shown that the typical trajectories of quantum mechanical particles are continuous but non-differentiable, and can be characterized by a fractal dimension which jumps from $D_F = 1$ at large length-scales to $D_F = 2$ at small length-scales, the transition occuring about the de Broglie scale (see refs [8, 9]).

Now such a fractal dimension $D_F = 2$ plays a particular role in physics. It is well-known that this is the fractal dimension of Brownian motion [10], *i.e.* from the mathematical view-point, of a Markov-Wiener process. This remark leads us to recall a related attempt at understanding the quantum behaviour, namely, Nelson's stochastic quantum mechanics [11, 12]. In this approach, it is assumed that any particle is subjected to an underlying Brownian motion of unknown origin, which is described by two (forward and backward) Wiener processes: when combined together they yield the complex nature of the wave function and they transform Newton's equation of dynamics into the Schrödinger equation.

This is precisely one of the aims of the Scale Relativity Theory (SRT) to relate the fractal and stochastic approaches [1, 8, 9, 13]: the hypothesis that the space-time is non-differentiable and fractal implies that there are an infinity of geodesics between any couple of points [8] and provides us with a fundamental and universal origin for the double Wiener process of Nelson [9].

SRT is a new approach to understand quantum mechanics, and moreover physical domains involving scale laws, such as chaotic systems. It is based on a generalization of Einstein's principle of relativity to scale transformations. Namely, one redefines space-time resolutions as characterizing the state of scale of reference systems, in the same way as speed characterizes their state of motion. Then one requires that the laws of physics apply whatever the state of the reference system, of motion (principle of motion-relativity) and of scale (principle of SRT). The principle of SRT is mathematically achieved by the principle of scale-covariance, requiring that the equations of physics keep their simplest form under transformations of resolution. In such conjecture, it was demonstrated that, in the topological dimension $D_T = 2$, the geodesics of the space-time are given by a Schrödinger's type equation [9–13]. Moreover, a hydrodynamic model was developed [14].

In the present paper, the Nottale's method is extended both for an arbitrary fractal dimension and for high order terms of the differential equation of motion. Then, in a particular type of motion, the complex speed potential simultaneously becomes wave function and a generalized Schrödinger type equation results.

2. MATHEMATICAL MODEL

Let us suppose that the motion of microparticles takes place on continuous but non-differentiable curves *i.e.* on fractals. A space-time manifold compatible with such motions will be called fractal space-time. According to SR method [1, 8, 9, 13], the non-differentiability implies forward ($dt > 0$) and backward ($dt > 0$) processes through the relations

$$d\boldsymbol{X}_+ = \boldsymbol{v}_+ dt + d\boldsymbol{\xi}_+, \quad d\boldsymbol{X}_- = \boldsymbol{v}_- dt + d\boldsymbol{\xi}_- \tag{1a,b}$$

where $\boldsymbol{X}$ is position vector of the microparticles, $\boldsymbol{v}_\pm$ are the forward and backward mean velocities,

$$\begin{aligned} \boldsymbol{v}_+ &= \frac{d\boldsymbol{x}_+}{dt} = \lim_{\Delta t \to 0+} \left\langle \frac{\boldsymbol{X}(t+\Delta t) - \boldsymbol{X}(t)}{\Delta t} \right\rangle, \\ \boldsymbol{v}_- &= \frac{d\boldsymbol{x}_-}{dt} = \lim_{\Delta t \to 0-} \left\langle \frac{\boldsymbol{X}(t) - \boldsymbol{X}(t-\Delta t)}{\Delta t} \right\rangle \end{aligned} \tag{2a,b}$$

and $d\boldsymbol{\xi}_{\pm}$ is a measure of non-differentiability with the property

$$\langle d\boldsymbol{\xi}_{\pm} \rangle = 0 \tag{3}$$

While the speed-concept is classically a single concept, if space-time is non-differentiable, we must introduce two speeds ($\boldsymbol{v}_+$ and $\boldsymbol{v}_-$) instead of one, even when going back to the classical domain. This "two-valuedness" of the speed vector is a new, specific consequence of non-differentiability that has no standard counterpart (in the sense of differential physics).

However, we cannot favor $\boldsymbol{v}_+$ rather than $\boldsymbol{v}_-$. The only solution is to consider both the forward ($dt > 0$) and backward ($dt < 0$) processes together. Than, we can use the complex speed [9, 13]:

$$\boldsymbol{V} = \frac{\boldsymbol{v}_+ + \boldsymbol{v}_-}{2} - i\frac{\boldsymbol{v}_+ - \boldsymbol{v}_-}{2} = \frac{d\boldsymbol{x}_+ + d\boldsymbol{x}_-}{2dt} - i\frac{d\boldsymbol{x}_+ - d\boldsymbol{x}_-}{2dt} \tag{4}$$

If $(\boldsymbol{v}_+ + \boldsymbol{v}_-)/2$ may be considered as differentiable (classical) speed, then the difference $(\boldsymbol{v}_+ - \boldsymbol{v}_-)/2$ is the non-differentiable speed.

Using the notations $d\boldsymbol{x}_{\pm} = d_{\pm}\boldsymbol{x}$, Eq.(4) becomes:

$$\boldsymbol{V} = \left(\frac{d_+ + d_-}{2dt} - i\frac{d_+ - d_-}{2dt} \right)\boldsymbol{x} \tag{5}$$

This enables us to define the operator:

$$\frac{\delta}{dt} = \frac{d_+ + d_-}{2dt} - i\frac{d_+ - d_-}{2dt} \tag{6}$$

Let us now assume that the curve describing the movement (continuous but non-differentiable) is immersed in a 3-dimensional space, and that $\boldsymbol{X}$ of components, X^i $(i = \overline{1,3})$ is the position vector of a point on the curve. Let us also consider a function $f(\boldsymbol{X}, t)$ and expand its total differential to the third order. We get:

$$df = \frac{\partial f}{\partial t}dt + \nabla f \cdot d\boldsymbol{X} + \frac{1}{2}\frac{\partial^2 f}{\partial X^i \partial X^j}dX^i dX^j + \frac{1}{6}\frac{\partial^3 f}{\partial X^i \partial X^j \partial X^k}dX^i dX^j dX^k + \ldots \tag{7}$$

From here, the forward and backward average values of this relation, using notations $dX_{\pm}{}^i = d_{\pm}X^i$, take the form:

$$\langle d_{\pm} f \rangle = \left\langle \frac{\partial f}{\partial t}dt \right\rangle + \langle \nabla f \cdot d_{\pm}\boldsymbol{X} \rangle + \frac{1}{2}\left\langle \frac{\partial^2 f}{\partial X^i \partial X^j}dX^i dX^j \right\rangle + \\ + \frac{1}{6}\left\langle \frac{\partial^3 f}{\partial X^i \partial X^j \partial X^k} d_{\pm}X^i d_{\pm}X^j d_{\pm}X^k \right\rangle + \ldots \tag{8}$$

We make the following stipulations: the mean values of the function f and its derivates coincide with themselves, and the differentials $d_{\pm}X^i$ and dt are independent, therefore the averages of their products coincide with the product of average. Thus Eq. (8) becomes:

$$d_{\pm}f = \frac{\partial f}{\partial t}dt + \nabla f\left\langle d_{\pm}\boldsymbol{X}\right\rangle + \frac{1}{2}\frac{\partial^2 f}{\partial X^i \partial X^j}\left\langle d_{\pm}X^i d_{\pm}X^j\right\rangle + \\ + \frac{1}{6}\frac{\partial^3 f}{\partial X^i \partial X^j \partial X^k}\left\langle d_{\pm}X^i d_{\pm}X^j d_{\pm}X^k\right\rangle \tag{9}$$

so that, further using Eqs. (1a,b) with the property (3),

$$d_{\pm}f = \frac{\partial f}{\partial t}dt + \nabla f d_{\pm}\boldsymbol{x} + \frac{1}{2}\frac{\partial^2 f}{\partial X^i \partial X^j}\left(d_{\pm}x^i d_{\pm}x^j + \left\langle d\xi_{\pm}^i d\xi_{\pm}^l\right\rangle\right) + \\ + \frac{1}{6}\frac{\partial^3 f}{\partial X^i \partial X^j \partial X^k}\left(d_{\pm}x^i d_{\pm}x^j d_{\pm}x^k + \left\langle d\xi_{\pm}^i d\xi_{\pm}^l d\xi_{\pm}^k\right\rangle\right). \tag{10}$$

Since $d\xi_{\pm}^i$ describes the fractal properties of the fractal curve which has the fractal dimension D_F (for details see [4, 9, 10]), it is natural to impose $\left(d\xi_{\pm}^i\right)^{D_F}$ to be proportional to time parameter dt, *i.e.*

$$\left(d\xi_{\pm}^i\right)^{D_F} \sim dt. \tag{11}$$

Even the average value of the fractal coordinate, $d\xi_{\pm}^i$, is null (see (3)) for the high order of the fractal coordinate average the situation can be different. For example, let us consider the means $\left\langle d\xi_{\pm}^i d\xi_{\pm}^j\right\rangle$ and $\left\langle d\xi_{\pm}^i d\xi_{\pm}^j d\xi_{\pm}^k\right\rangle$, respectively.

Firstly, let us focus on the mean $\left\langle d\xi_{\pm}^i d\xi_{\pm}^j\right\rangle$. If $i \neq j$ this average is zero due the independence of $d\xi^i$ and $d\xi^j$. So, using (11) we can write:

$$\left\langle d\xi_{\pm}^i d\xi_{\pm}^j\right\rangle = \delta^{ij}(2Ddt)^{2/D_F}, \tag{12a}$$

with

$$\delta^{ij} = \begin{cases} 1, & \text{if} \quad i = j \\ 0, & \text{if} \quad i \neq j \end{cases}$$

and D an adequate constant of proportionality.

Particularly, through of a Peano type curves, which covers a two-dimensional surface, *i.e.* $D_F = 2$, (12a) becomes:

$$\left\langle d\xi_{\pm}^i d\xi_{\pm}^j\right\rangle = \pm\delta^{ij}(2Ddt)$$

where we had considered that:

$$\begin{cases}\left\langle d\xi_+^i d\xi_+^l \right\rangle > 0 \quad \text{and} \quad dt > 0 \\ \left\langle d\xi_-^i d\xi_-^l \right\rangle > 0 \quad \text{and} \quad dt < 0.\end{cases}$$

Now, let us focus on the mean, $\left\langle d\xi_\pm^i d\xi_\pm^j d\xi_\pm^k \right\rangle$. If $i \neq j \neq k$ this average is zero due the independence of $d\xi^i$ from $d\xi^j$ and $d\xi^k$. Therefore, using Eq. (11) and an adequate choice of the constant of proportionality, $6D^2/c$, we can write:

$$\left\langle d\xi_\pm^i d\xi_\pm^j d\xi_\pm^k \right\rangle = \delta^{ijk} \frac{6D^2}{c} (dt)^{3/D_F}, \tag{12b}$$

with

$$\delta^{ijk} = \begin{cases} 1, & \text{if} \quad i = j = k \\ 0, & \text{if} \quad i \neq j \neq k. \end{cases}$$

Particularly, through a generalization of Peano type curves which cover a three-dimensional cube *i.e.* D_F asymptotically tend to the topological dimension [8, 9] $D_T = 3$, the Eq. (12b) becomes:

$$\left\langle d\xi_\pm^i d\xi_\pm^j d\xi_\pm^k \right\rangle = \delta^{ijk} \left(\frac{6D^2}{c} dt \right),$$

where we considered that:

$$\begin{cases}\left\langle d\xi_+^i d\xi_+^j d\xi_+^k \right\rangle > 0 \quad \text{and} \quad dt > 0 \\ \left\langle d\xi_-^i d\xi_-^j d\xi_-^k \right\rangle < 0 \quad \text{and} \quad dt < 0.\end{cases}$$

Then Eq. (10) may be written under the form:

$$\begin{aligned} d_\pm f = {} & \frac{\partial f}{\partial t} dt + \nabla f d_\pm \boldsymbol{x} + \frac{1}{2} \frac{\partial^2 f}{\partial X^i \partial X^j} d_\pm x^i d_\pm x^j + \frac{1}{2} \frac{\partial^2 f}{\partial X^i \partial X^j} \delta^{ij} 2D (dt)^{(2/D_F)} + \\ & + \frac{1}{6} \frac{\partial^3 f}{\partial X^i \partial X^j \partial X^k} d_\pm x^i d_\pm x^j d_\pm x^k + \frac{\partial^3 f}{\partial X^i \partial X^j \partial X^k} \delta^{ijk} \frac{D^2}{c} (dt)^{(3/D_F)}. \end{aligned} \tag{13}$$

If we divide by *dt* and neglect the terms which contain differential factors (for details on the method see [8, 9, 13], Eq. (13) is reduced to:

$$\frac{d_\pm f}{dt} = \frac{\partial f}{\partial t} + \boldsymbol{v}_\pm \nabla f \pm D (dt)^{(2/D_F)-1} \Delta f + \frac{D^2}{c} (dt)^{(3/D_F)-1} \nabla^3 f, \tag{14}$$

with

$$\nabla^2 f = \frac{\partial^2 f}{(\partial X^1)^2} + \frac{\partial^2 f}{(\partial X^2)^2} + \frac{\partial^2 f}{(\partial X^3)^2},, \qquad \nabla^3 f = \frac{\partial^3 f}{(\partial X^1)^3} + \frac{\partial^3 f}{(\partial X^2)^3} + \frac{\partial^3 f}{(\partial X^3)^3}.$$

This relation also allows us to give the definition of the operator

$$\frac{d_\pm}{dt} = \frac{\partial}{\partial t} + \boldsymbol{v}_\pm \nabla \pm D(dt)^{(2/D_F)-1}\Delta + \frac{D^2}{c}(dt)^{(3/D_F)-1}\nabla^3. \tag{15}$$

Let us calculate, under the circumstances $(\delta f / dt)$. Taking into account Eqs. (15), and Eq. (6), we obtain:

$$\begin{aligned}
\frac{\delta f}{dt} &= \frac{1}{2}\left[\frac{d_+ f}{dt} + \frac{d_- f}{dt} - i\left(\frac{d_+ f}{dt} - \frac{d_- f}{dt}\right)\right] = \\
&= \frac{1}{2}\left(\frac{\partial f}{\partial t} + \boldsymbol{v}_+ \nabla f + D(dt)^{(2/D_F)-1}\Delta f + \frac{D^2}{c}(dt)^{(3/D_F)-1}\nabla^3 f\right) + \\
&+ \frac{1}{2}\left(\frac{\partial f}{\partial t} + \boldsymbol{v}_- \nabla f - D(dt)^{(2/D_F)-1}\Delta f + \frac{D^2}{c}(dt)^{(3/D_F)-1}\nabla^3 f\right) - \\
&- \frac{i}{2}\left[\left(\frac{\partial f}{\partial t} + \boldsymbol{v}_+ \nabla f + D(dt)^{(2/D_F)-1}\Delta f + \frac{D^2}{c}(dt)^{(3/D_F)-1}\nabla^3 f\right) - \right. \\
&\left. - \left(\frac{\partial f}{\partial t} + \boldsymbol{v}_- \nabla f - D(dt)^{(2/D_F)-1}\Delta f + \frac{D^2}{c}(dt)^{(3/D_F)-1}\nabla^3 f\right)\right] \\
&= \frac{\partial f}{\partial t} + \left(\frac{\boldsymbol{v}_+ + \boldsymbol{v}_-}{2} - i\frac{\boldsymbol{v}_+ - \boldsymbol{v}_-}{2}\right)\nabla f - iD(dt)^{(2/D_F)-1}\Delta f + \frac{D^2}{c}(dt)^{(3/D_F)-1}\nabla^3 f,
\end{aligned} \tag{16}$$

or using Eq. (5):

$$\frac{\delta f}{dt} = \frac{\partial f}{\partial t} + \boldsymbol{V} \cdot \nabla f - iD(dt)^{(2/D_F)-1}\Delta f + \frac{D^2}{c}(dt)^{(3/D_F)-1}\nabla^3 f. \tag{17}$$

This relation also allows us to give the definition of the non-differentiable operator:

$$\frac{\delta}{dt} = \frac{\partial}{\partial t} + \boldsymbol{V} \cdot \nabla - iD(dt)^{(2/D_F)-1}\Delta + \frac{D^2}{c}(dt)^{(3/D_F)-1}\nabla^3. \tag{18}$$

We now apply the principle of scale covariance, and postulate that the passage from classical (differentiable) mechanics to the non-differentiable mechanics which is considered here can be implemented by replacing the standard time derivative *d/dt* by the complex operator, δ / dt (this results in a generalization of the principle of scale covariance given by Nottale in [8, 9, 13]). As a consequence, we are now able to write the equation of geodesics (a generalization of the first Newton's principle) in a non-differentiable space-time under its covariant form:

$$\frac{\delta \boldsymbol{V}}{dt} = \frac{\partial \boldsymbol{V}}{\partial t} + \boldsymbol{V}\cdot\nabla\boldsymbol{V} - iD\left(dt\right)^{(2/D_F)-1}\Delta\boldsymbol{V} + \frac{D^2}{c}\left(dt\right)^{(3/D_F)-1}\nabla^3\boldsymbol{V} = 0 \tag{19}$$

i.e. a Burgers – Korteweg – de Vries (BKdV) type equation in a fractal space-time. This means that the complex acceleration field, $\delta\boldsymbol{V}/dt$, depends on the local time dependence, $\partial_t\boldsymbol{V}$, on the non-linearity (convective) term, $\boldsymbol{V}\cdot\nabla\boldsymbol{V}$, on the dissipative term $\Delta\boldsymbol{V}$, and on the dispersive one, $\nabla^3\boldsymbol{V}$. Moreover, the behaviour of a "non-differentiable fluid" is of viscoelastic or of hysteretic type which means that the fractal space-time has memory. Such a result is in agreement with the opinion given in [15–17]: the non-differentiable fluid can be described by Kelvin-Voight or Maxwell rheological model with the aid of complex quantities *e.g.* the complex speed field, the complex acceleration field or the complex coefficient structure etc.

From Eq. (19) and through the operational relation $\boldsymbol{V}\cdot\nabla\boldsymbol{V} = \nabla(\boldsymbol{V}^2/2) - \boldsymbol{V}\times(\nabla\times\boldsymbol{V})$ we obtain the equation:

$$\frac{\delta \boldsymbol{V}}{dt} = \frac{\partial \boldsymbol{V}}{\partial t} + \nabla\left(\frac{\boldsymbol{V}^2}{2}\right) - \boldsymbol{V}\times\left(\nabla\times\boldsymbol{V}\right) - iD\left(dt\right)^{(2/D_F)-1}\Delta\boldsymbol{V} + \frac{D^2}{c}\left(dt\right)^{(3/D_F)-1}\nabla^3\boldsymbol{V} = 0. \tag{20}$$

If the motions of the "non-differentiable fluid" are irrotational, *i.e.* $\boldsymbol{\Omega} = \nabla\times\boldsymbol{V} = 0$ we can choose $\boldsymbol{V}$ of the form:

$$\boldsymbol{V} = \nabla\phi, \tag{21}$$

with ϕ a complex speed potential. Then, Eq. (20) becomes:

$$\frac{\delta \boldsymbol{V}}{dt} = \frac{\partial \boldsymbol{V}}{\partial t} + \nabla\left(\frac{\boldsymbol{V}^2}{2}\right) - iD\left(dt\right)^{(2/D_F)-1}\Delta\boldsymbol{V} + \frac{D^2}{c}\left(dt\right)^{(3/D_F)-1}\nabla^3\boldsymbol{V} = 0 \tag{22}$$

and more, by substituting Eq. (21) in Eq. (22) and integrating, we shall have by integration

$$\frac{\partial\phi}{\partial t} + \frac{1}{2}\left(\nabla\phi\right)^2 - iD\left(dt\right)^{(2/D_F)-1}\Delta\phi + \frac{D^2}{c}\left(dt\right)^{(3/D_F)-1}\nabla^3\phi = F(t), \tag{23}$$

with $F(t)$ being a function of time only. We realize that Eq. (22) has been reduced to a single scalar relation (23), *i.e.* a Bernoulli-type equation.

Let us choice the complex speed potential in the form:

$$\phi = -2iD\ln\psi. \tag{24}$$

Then, according to [8, 9, 13], ψ becomes a wave-function and satisfies, by means of Eq. (23) and up to an arbitrary phase factor which may be set to zero by a suitable choice of the phase of ψ, $F(t)\equiv 0$, a Schrödinger type equation:

$$D^2\Delta\psi + iD\partial_t\psi + \frac{U}{2m}\psi = 0, \tag{25}$$

with U the complex potential,

$$U = 2imD^2\left(-\Delta\psi + (dt)^{(2/D_F)-1}\Delta\psi + \frac{D}{c}(dt)^{(3/D_F)-1}\nabla^3\ln\psi\right) \tag{26}$$

and m the rest mass of a 'test particle'.

3. CONCLUSIONS

The main conclusions of the present paper are the following: i) a non-differentiable continuum is necessarily fractal and the trajectories in such a space (or space-time) own (at least) the following properties: i1) the test particle can follow an infinity of potential trajectories: this leads one to use a fluid-like description; i2) the geometry of each trajectory is fractal (of dimension D_F). Each elementary displacement is then described in terms of the sum, $d\boldsymbol{X} = d\boldsymbol{x} + d\boldsymbol{\xi}$, of a mean classical displacement $d\boldsymbol{x} = \boldsymbol{v}dt$ and of a fractal fluctuation $d\boldsymbol{\xi}$, whose behavior satisfies the principle of SRT (in its simplest Galilean version). It is such that $\langle d\boldsymbol{\xi}\rangle = 0$, $\langle d\boldsymbol{\xi}^2\rangle = 2D(dt)^{(2/D_F)-1}$ and $\langle d\boldsymbol{\xi}^3\rangle = (D^2/c)(dt)^{(3/D_F)-1}$ where D defines the fractal/non-fractal transition, *i.e.* the transition from the explicit scale dependence to scale independence. The existence of this fluctuation implies introducing new terms [18] (of second and third order) in the differential equation of motion; 13) time reversibility is broken at the infinitesimal level: this can be described in terms of a two-valuedness of the velocity vector for which we use a complex representation, $\boldsymbol{V} = (\boldsymbol{v}_+ + \boldsymbol{v}_-)/2 - i(\boldsymbol{v}_+ - \boldsymbol{v}_-)/2$, with $\boldsymbol{v}_+$ the "forward" speed and $\boldsymbol{v}_-$ the "backward" speed; ii) In such context, a Burgers – Korteweg – de Vries (BKdV) equation is obtained. If the motions of the "non-differentiable fluid" are irrotational, the BKdV type equation is reduced to a non-linear Schrödinger type equation. In this case, the scalar complex velocity field simultaneously becomes wave function.

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