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SOME ASPECTS OF ELECTROMAGNETIC MULTIPOLE EXPANSIONS

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Abstract. Various procedures for expressing the multipolar expansion of the electromagnetic field are considered with application to the calculation of the radiated power. Some results from the literature are discussed.

Key words: electromagnetic multipoles, toroidal moments, radiated power.

1. INTRODUCTION

The multipole expansion of the electromagnetic field is an useful tool in physics and it can be found in any book on electrodynamics or on the theory of atomic and nuclear transitions. In most of the textbooks (see, *e.g.* [1, 2]) a full and systematic treatment is given in spherical coordinates, while for Cartesian coordinates the problem is presented completely only for the static case. The dynamic case is given only for the lowest order multipoles. A general procedure for the reduction of the multipole tensors to fully symmetric traceless ones in the static case is given in [3, 4] and it is generalized in [5] to the dynamic case. This method is applied in [6] to the radiation field.

An interesting result of the reduction procedure is the natural appearance of the toroid moments, a third family of multipole moments. A model of a toroid dipole that generates all other moments of the family by known rules is a conventional solenoid folded into a torus. Dubovik and Cheshkov published a summarizing paper in 1974 [10] and Dubovik with Tugushev a review in 1990 [12], presenting the importance of the toroid moments in electrodynamics, solid-state physics, atomic and nuclear physics, elementary particles, *i.e.* almost everywhere.

In the present paper different procedures are applied for calculating the total power radiated by a confined system of charges. A first procedure represents the traditional method of expressing the field expansions by the multipole Cartesian tensors, applying finally the reduction of these tensors. The second procedure uses the reduction technique done in [5], but in more systematic way considered here. The results of [6] are analyzed and compared to those of other methods from the literature [8, 9]. The advantages of the Cartesian coordinates are emphasized firstly by the simplicity of the formalism: only algebraic manipulations and combinatorics are implied, no special functions being required. Secondly, the procedure initiated in [5] and used here leads to a nontrivial grouping of different multipolar contributions standing out the toroidal multipole contributions [10–12].

In section 2, some basic formulas for multipole expansions are presented. Section 3 deals with the radiation field as well as with the expression and expansion of the total radiated power. In Section 4, the procedure for reduction of a tensor to a symmetric traceless one is given. The total radiated power is then treated in Section 5 using the reduced moments and a comparison with literature is made. Section 6 presents, in a systematic and concise way, the results from [5]. Then, in Section 7, the total radiated power is expressed with the help of transformed moments. The conclusions are given in Section 8.

2. BASIC FORMULAS FOR THE MULTIPOLE EXPANSIONS

Let us consider charge $\rho(\boldsymbol{r}, t)$ and current $\boldsymbol{j}(\boldsymbol{r}, t)$ distributions having supports included in a finite domain $\mathcal{D}$. Choosing the origin O of the Cartesian coordinates in $\mathcal{D}$, and using the notation $\boldsymbol{e}_i$ for the orthogonal unit vectors along the axes, the retarded scalar and vector potentials at a point outside $\mathcal{D}$, $\boldsymbol{r} = x_i \boldsymbol{e}_i$, are

$$\Phi(\boldsymbol{r}, t) = \frac{1}{4\pi\varepsilon_0} \int \frac{\rho(\boldsymbol{\xi}, t - R/c)}{R} \mathrm{d}^3\xi, \quad \boldsymbol{A}(\boldsymbol{r}, t) = \frac{\mu_0}{4\pi} \int \frac{\boldsymbol{j}(\boldsymbol{\xi}, t - R/c)}{R} \mathrm{d}^3\xi, \tag{1}$$

where $\boldsymbol{R} = r - \boldsymbol{\xi}$. The Taylor series expansion of the function $f(R)$ is

$$f(R) = \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} \xi_{i_1} \ldots \xi_{i_n} \partial_{i_1 \ldots i_n} f(r) = \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} \boldsymbol{\xi}^n \,||\, \nabla^n f(r),$$

where $\boldsymbol{a}^n$ is the n-fold tensorial product $\boldsymbol{a} \otimes \ldots \otimes \boldsymbol{a}$: $(\boldsymbol{a} \otimes \ldots \otimes \boldsymbol{a})_{i_1 \ldots i_n} = a_{i_1} \ldots a_{i_n}$. Denoting by $\mathsf{T}^{(n)}$ a n-th order tensor, $\mathsf{A}^{(n)} \,||\, \mathsf{B}^{(m)}$ is a $|n - m|$-th order tensor with the components:

$$(\mathbf{A}^{(n)} \,||\, \mathbf{B}^{(m)})_{i_1 \cdots i_{|n-m|}} = \begin{cases} A_{i_1 \cdots i_{n-m} j_1 \cdots j_m} B_{j_1 \cdots j_m}, & n > m \\ A_{j_1 \cdots j_n} B_{j_1 \cdots j_n}, & n = m \\ A_{j_1 \cdots j_n} B_{j_1 \cdots j_n i_1 \cdots i_{m-n}}, & n < m \end{cases}$$

By applying the formula for the Taylor series expansion to the scalar potential we get:

$$\Phi(\boldsymbol{r}, t) = \frac{1}{4\pi\varepsilon_0} \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} \nabla^n \,||\left[\frac{\mathbf{P}^{(n)}(t_0)}{r}\right], \quad t_0 = t - \frac{r}{c}, \tag{2}$$

$\mathbf{P}^{(n)}$ being the n-th order electric multipole tensor

$$\mathbf{P}^{(n)}(t) = \int_{\mathcal{D}} r^n \rho(r, t) \mathrm{d}^3 x. \tag{3}$$

For the vector potential we obtain the expression:

$$\boldsymbol{A}(r, t) = \frac{\mu_0}{4\pi} \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} \nabla^n \,||\left[\frac{\boldsymbol{\mu}^{(n+1)}(t_0))}{r}\right]. \tag{4}$$

In the previous equation the magnetic multipole tensor was introduced by its Cartesian components:

$$\mu_{i_1 \cdots i_n i}(t) = \int_{\mathcal{D}} x_{i_1} \ldots x_{i_n} j_i(r, t) \mathrm{d}^3 x.$$

3. THE RADIATION FIELD

The formula for the power radiated by a charged system described by the charge ρ and current $\boldsymbol{j}$ densities with supports included in a finite domain $\mathcal{D}$ is well known [2, 4]:

$$\mathcal{J}(\mathbf{v}) = \frac{\mathrm{d}P}{\mathrm{d}\Omega}(\mathbf{v}, t) = \frac{r^2}{\mu_0 c}\left[\mathbf{v} \times \frac{\partial}{\partial t} \boldsymbol{A}_{rad}(\boldsymbol{r}, t)\right]^2. \tag{5}$$

Here, $\mathbf{v} = \boldsymbol{r}/r$ and $\mathrm{d}P/\mathrm{d}\Omega$ is related to the flow of the energy detected in the observation point $\boldsymbol{r}$, at large distance r compared with the dimensions of the given charged system. The vector $\boldsymbol{A}_{rad}$ is obtained from the retarded potential (1) by retaining only the dominant terms at large distances.

In the following, the expansion of $\boldsymbol{A}_{rad}$ is derived [4]. Starting from:

$$\boldsymbol{A}(\boldsymbol{r},t) \approx \frac{\mu_0}{4\pi}\frac{1}{r}\sum_{n=0}^{\infty}\frac{(-1)^n}{n!}\nabla^n \parallel \boldsymbol{\mu}^{(n+1)}(t_0),$$

one obtains the following expression for the part of $\boldsymbol{A}$ contributing to the radiation:

$$\boldsymbol{A}_{rad}(\boldsymbol{r},t) = \frac{\mu_0}{4\pi}\frac{1}{r}\sum_{n\geq 0}\frac{1}{n!c^n}[\boldsymbol{\nu}^n \parallel \boldsymbol{\mu}^{(n+1)}_{,n}(t_0)]; \quad \boldsymbol{\mu}^{(n+1)}_{,n} = \frac{\mathrm{d}^n}{\mathrm{d}t^n}\boldsymbol{\mu}^{(n+1)}.$$

Considering the angular distribution of the radiation given by equation (5) and applying the expansion of $\boldsymbol{A}_{rad}$, one gets:

$$\mathcal{J}(\boldsymbol{\nu}) = \frac{1}{16\pi^2\varepsilon_0 c^3}\left[\boldsymbol{\nu}\times\sum_{n\geq 0}\frac{1}{n!c^n}\left[\boldsymbol{\nu}^n \parallel \boldsymbol{\mu}^{(n+1)}_{,n+1}\right]\right]^2.$$

Finally, the following result for the angular distribution of the radiation is obtained:

$$16\pi^2\varepsilon_0 c^3\mathcal{J}(\boldsymbol{\nu}) = \sum_{n\geq 0, m\geq 0}\frac{1}{n!m!c^{n+m}}\left[\left(\boldsymbol{\nu}^n \parallel \boldsymbol{\mu}^{(n+1)}_{,n+1}\right)_k\left(\boldsymbol{\nu}^m \parallel \boldsymbol{\mu}^{(m+1)}_{,m+1}\right)_k - \right.$$
$$\left. -\left(\boldsymbol{\nu}^{n+1} \parallel \boldsymbol{\mu}^{(n+1)}_{,n+1}\right)\left(\boldsymbol{\nu}^{m+1} \parallel \boldsymbol{\mu}^{(m+1)}_{,m+1}\right)\right]. \tag{6}$$

In order to calculate the total radiated power, the formula

$$\mathcal{J} = 4\pi < \mathcal{J}(\boldsymbol{\nu}) >$$

is used, where

$$< f(\boldsymbol{\nu}) > = (1/4\pi)\int f(\boldsymbol{\nu})\mathrm{d}\Omega(\boldsymbol{\nu}).$$

In the previous equation we need the averaging formula:

$$< \nu_{i_1}\ldots\nu_{i_{2n-1}}\nu_{i_{2n}} > = \frac{1}{(2n+1)!!}\sum_{D(i)}\delta_{i_1 i_2}\ldots\delta_{i_{2n-1}i_{2n}}, \tag{7}$$

where $D(i)$ symbolizes the sum over all permutations of the indices $i_1, i_2, \ldots, i_n$ which give distinct terms.

When working with the expansion of the radiated power, one must have a strict criterion regarding the comparison of different terms contributions. This criterion can be easily obtained if one refers, particularly, to monochromatic sources. If the possibility to represent any type of variation in time as a superposition of functions corresponding to the monochromatic sources is taken into account, the conclusions will be general. In the case of the time variation characterized by the pulsation ω, the magnetic moment contributes with a term of the order d/λ, *i.e.* $\mu \sim d/\lambda$. We can write

$$\frac{\mathrm{d}^k}{\mathrm{d}t^k}\mu^{(n)}(t) \sim \begin{cases} \left(\frac{d}{\lambda}\right)^{k+1}, & k \le n \\ \left(\frac{d}{\lambda}\right)^{n+1}, & k \ge n \end{cases}, \quad \frac{\mathrm{d}^k}{\mathrm{d}t^k}\mathbf{P}^{(n)}(t) \sim \begin{cases} \left(\frac{d}{\lambda}\right)^{k}, & k \le n \\ \left(\frac{d}{\lambda}\right)^{n}, & k \ge n. \end{cases} \tag{8}$$

Because the expansion of the radiated power comes from the values of the field in the region $r > \lambda > d$, it is obvious that the order of magnitude of a term is defined by the sum $n + m$. Therefore, a consistent way of using a finite number of terms in the expansion (6), terms that accurately characterize the radiation up to a given order M, is to retain all the terms corresponding to the values $n + m$ between 0 and M. Such a procedure was not used consistently everywhere in literature, as it will be shown later on. On the other hand, from equation (6), regarding the radiated power, we may write in the case of monochromatic sources,

$$\mathcal{J} = \sum_{M=1}^{\infty} \mathcal{J}^{(M)} = \sum_{n+m=M} \mathcal{J}^{(n,m)} = \left(\frac{d}{\lambda}\right)^2 \sum_{n=0}^{\infty} a_{2n}\left(\frac{d}{\lambda}\right)^{2n}, \tag{9}$$

with

$$4\pi\varepsilon_0 \mathcal{J}^{(n,m)} = \frac{1}{n!m!c^{n+m+3}}\Big\langle \left(\mathbf{\nu}^n \parallel \boldsymbol{\mu}^{(n+1)}_{,n+1}\right)_k \left(\mathbf{\nu}^m \parallel \boldsymbol{\mu}^{(m+1)}_{,m+1}\right)_k - \\ -\left(\mathbf{\nu}^{n+1} \parallel \boldsymbol{\mu}^{(n+1)}_{,n+1}\right)\left(\mathbf{\nu}^{m+1} \parallel \boldsymbol{\mu}^{(m+1)}_{,m+1}\right)\Big\rangle. \tag{10}$$

Below, considering the order of magnitude for different terms from the expansion of the total radiated power, we consider only the terms from the last sum in the right hand side of equation (9). Therefore, the order of a term is specified by the sum $n + m$.

For particular cases,

$$\mathcal{J}^{(0)} = \mathcal{J}^{(0,0)}, \quad \mathcal{J}^{(1)} = \mathcal{J}^{(0,1)} + \mathcal{J}^{(1,0)} = 0,$$
$$\mathcal{J}^{(2)} = \mathcal{J}^{(0,2)} + \mathcal{J}^{(2,0)} + \mathcal{J}^{(1,1)}, \quad \mathcal{J}^{(3)} = 0,$$
$$\mathcal{J}^{(4)} = \mathcal{J}^{(2,2)} + \mathcal{J}^{(1,3)} + \mathcal{J}^{(3,1)} + \mathcal{J}^{(0,4)} + \mathcal{J}^{(4,0)},$$

where the fact that the averaged terms with $n + m$ odd are zero is taken into account.

We present the first terms of the expansion:

$$4\pi\varepsilon_0 c^3 \mathcal{J}^{(0,0)} = \langle \dot{\boldsymbol{\mu}}\dot{\boldsymbol{\mu}} - (\mathbf{\nu}\cdot\dot{\boldsymbol{\mu}})(\mathbf{\nu}\cdot\dot{\boldsymbol{\mu}})\rangle = \frac{2}{3}\dot{\boldsymbol{\mu}}^2 \sim \left(\frac{d}{\lambda}\right)^2$$
$$4\pi\varepsilon_0 c^3 \mathcal{J}^{(1,1)} = \frac{1}{15c^2}\left[4\ddot{\mu}_{ij}\ddot{\mu}_{ij} - \ddot{\mu}_{ij}\ddot{\mu}_{ji} - \ddot{\mu}_{ii}\ddot{\mu}_{jj}\right] \sim \left(\frac{d}{\lambda}\right)^4 \tag{11}$$
$$4\pi\varepsilon_0 c^3 \mathcal{J}^{(0,2)} = 4\pi\varepsilon_0 c^3 \mathcal{J}^{(2,0)} = \frac{1}{15c^2}\left[2\dot{\mu}_i\dddot{\mu}_{jji} - \dot{\mu}_i\dddot{\mu}_{ijj}\right] \sim \left(\frac{d}{\lambda}\right)^4.$$

The expressions for $\mathcal{J}^{(0,0)}$, $\mathcal{J}^{(1,1)}$, $\mathcal{J}^{(0,2)}$, $\mathcal{J}^{(2,0)}$ are given by Bellotti and Bornatici in [8]. They go further, introducing the reduced multipole moments and finding a new term, as we will see in the following.

The standard magnetic moments are defined as:

$$\mathbf{M}^{(n)} = \frac{n}{n+1}\int_{\mathcal{D}} \boldsymbol{\xi}^n \times \boldsymbol{j}\,\mathrm{d}^3\xi : \mathbf{M}_{i_1\ldots i_n} = \frac{n}{n+1}\int_{\mathcal{D}} \xi_{i_1}\ldots\xi_{i_{n-1}}(\boldsymbol{\xi}\times\boldsymbol{j})_{i_n}\,\mathrm{d}^3\xi. \tag{12}$$

It can be shown that instead of the expression (4) one can use an expansion obtained from it by performing the substitution:

$$\mu_{i_1\ldots i_n} \longrightarrow -\varepsilon_{i_n i_{n-1}k}\mathbf{M}_{i_1\ldots i_{n-2}k} + \frac{1}{n}\dot{\mathbf{P}}_{i_1\ldots i_n}. \tag{13}$$

The result for the vector potential is given in [4, 5]:

$$\boldsymbol{A}(\boldsymbol{r},t) = \frac{\mu_0}{4\pi}\boldsymbol{\nabla}\times\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{n!}\boldsymbol{\nabla}^{n-1}\parallel\left[\frac{1}{r}\mathbf{M}^{(n)}(t_0)\right] + \frac{\mu_0}{4\pi}\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{n!}\boldsymbol{\nabla}^{n-1}\parallel\left[\frac{1}{r}\dot{\mathbf{P}}^{(n)}(t_0)\right].$$

Using the transformation (13) in the equations (11) and performing the integration, one obtains:

$$4\pi\varepsilon_0 c^3\mathcal{J}^{(0,0)} = \dot{\mu}_i\dot{\mu}_i - <\nu_{i_1}\nu_{i_2}>\dot{\mu}_{i_1}\dot{\mu}_{i_2} = \frac{2}{3}\ddot{\boldsymbol{p}}^2,$$

$$4\pi\varepsilon_0 c^3\mathcal{J}^{(1,1)} = \frac{2}{3}\ddot{\boldsymbol{m}}^2 + \frac{1}{20c^2}\dddot{\mathbf{P}}_{ij}\dddot{\mathbf{P}}_{ij} - \frac{1}{60c^2}\dddot{\mathbf{P}}_{ii}\dddot{\mathbf{P}}_{jj},$$

$$4\pi\varepsilon_0 c^3\mathcal{J}^{(0,2)} = 4\pi\varepsilon_0 c^3\mathcal{J}^{(2,0)} = \frac{1}{2c^2}\left[-\frac{1}{3}\ddot{p}_i\dddot{\mathbf{N}}_i + \frac{2}{45}\ddot{p}_i\dddot{\mathbf{P}}_{iqq}\right]$$

$$4\pi\varepsilon_0 c^3\mathcal{J}^{(2,2)} = \frac{1}{4c^4}\left[\frac{1}{15}(4\dddot{\mathbf{M}}_{ij}\dddot{\mathbf{M}}_{ij} - \dddot{\mathbf{M}}_{ij}\dddot{\mathbf{M}}_{ji}) - \frac{2}{45}\mathbf{N}_i\ddddot{\mathbf{P}}_{qqi} - \right.$$
$$\left. -\frac{2}{9\times 105}\ddddot{\mathbf{P}}_{qqi}\ddddot{\mathbf{P}}_{qqi} + \frac{8}{9\times 105}\ddddot{\mathbf{P}}_{ijk}\ddddot{\mathbf{P}}_{ijk}\right],$$

$$4\pi\varepsilon_0 c^3\mathcal{J}^{(1,3)} = 4\pi\varepsilon_0\mathcal{J}^{(3,1)} = \frac{1}{6c^4}\left[\frac{4}{15}\ddot{\mathbf{M}}_k\dddot{\mathbf{M}}_{qqk} - \frac{1}{15}\mathbf{N}^{(3,1)}_{li}\dddot{\mathbf{P}}_{li} - \right.$$
$$\left. -\frac{3}{8\times 105}\dddot{\mathbf{P}}_{qq}\overset{\cdot\cdot\cdot\cdot\cdot}{\mathbf{P}}_{iijj} + \frac{9}{8\times 105}\dddot{\mathbf{P}}_{ij}\overset{\cdot\cdot\cdot\cdot\cdot}{\mathbf{P}}_{qqij}\right.,$$

$$4\pi\varepsilon_0 c^3\mathcal{J}^{(0,4)} = 4\pi\varepsilon_0 c^3\mathcal{J}^{(4,0)} = \frac{1}{24c^4}\left[-\frac{1}{5}\varepsilon_{ijk}\overset{\cdot\cdot\cdot\cdot\cdot}{\mathbf{M}}_{qqjk}\ddot{p}_i + \frac{2}{5\times 35}\ddot{p}_i\overset{\cdot\cdot\cdot\cdot\cdot\cdot}{\mathbf{P}}_{qqjji}\right],$$

where

$$\mathbf{N}_i = \varepsilon_{ips}\mathbf{M}_{ps} = \frac{2}{3}\int_{\mathcal{D}}[\boldsymbol{\xi}\times(\boldsymbol{\xi}\times\boldsymbol{j})]\mathrm{d}^3\xi, \qquad \mathbf{N}^{(3,1)}_{ij} = \varepsilon_{jps}\mathbf{M}_{ips}. \tag{14}$$

4. SYMMETRISING AND DETRACING TENSORS

In this section the symmetrisation and detracing method specific for the electromagnetic moments is presented. The procedure and the notation of [4] are used. The first step is to symmetrise the magnetic tensors $\mathbf{M}^{(n)}$, already symmetric in the first $n-1$ indices:

$$\mathbf{M}_{(sym)i_1\ldots i_n} = \frac{1}{n}[\mathbf{M}_{i_1\ldots i_n} + \mathbf{M}_{i_n\ldots i_1} + \ldots \mathbf{M}_{i_1\ldots i_n, i_{n-1}}] \equiv \frac{1}{n}\sum_{D(i)} \mathbf{M}_{i_1\ldots i_n}.$$

The expression can also be written as:

$$\mathbf{M}_{(sym)i_1\ldots i_n} = \mathbf{M}_{i_1\ldots i_n} - \frac{1}{n}\sum_{\lambda=1}^{n-1} \varepsilon_{i_\lambda i_n q} \mathbf{N}^{(n,1)(\lambda)}_{i_1\ldots i_{n-1}q},$$

where $T^{\ldots(\lambda)}_{i_1\ldots}$ is the component which does not have the i_λ index, while:

$$\mathbf{N}^{(n,1)}_{i_1\ldots i_{n-1}} = \varepsilon_{i_{n-1}ps} M_{i_1\ldots i_{n-2}ps}$$

is a $n-1$ order tensor. Further, the correspondence: $\mathbf{T}^{(n)} \to \mathcal{N}[\mathbf{T}^{(n)}]$ is introduced, where $\mathcal{N}[\mathbf{T}^{(n)}]$ is a $n-1$ order tensor:

$$\mathcal{N}[\mathbf{T}^{(n)}]_{i_1\ldots i_{n-1}} = \varepsilon_{i_{n-1}ps} \mathbf{T}_{i_1\ldots i_{n-2}ps}$$

and $\mathcal{N}^k[\mathbf{M}^{(n)}] \equiv \mathbf{N}^{(n,k)}$ is a tensor of rank $n-k$. Here are some examples:

$$\mathbf{N}^{(n,1)}_{i_1\ldots i_{n-1}} = \frac{n}{n+1}\int \xi_{i_1}\ldots\xi_{i_{n-2}}[\boldsymbol{\xi}\times(\boldsymbol{\xi}\times \boldsymbol{j})]_{i_{n-1}}\,\mathrm{d}^3\xi,$$

$$\mathbf{N}^{(n,2)}_{i_1\ldots i_{n-2}} = -\frac{n}{n+1}\int \xi^2\xi_{i_1}\ldots\xi_{i_{n-3}}(\boldsymbol{\xi}\times \boldsymbol{j})_{i_{n-2}}\,\mathrm{d}^3\xi.$$

Below, the following relations are required:

$$\mathcal{N}^{2k}[\mathbf{M}^{(n)}] \equiv \mathbf{N}^{(n;2k)} = \frac{(-1)^k n}{n+1}\int_{\mathcal{D}} \xi^{2k}\boldsymbol{\xi}^{n-2k}\times \boldsymbol{j}\mathrm{d}^3\xi = (-1)^k \mathbf{M}^{(n;k)},$$

$$\mathcal{N}^{2k+1}[\mathbf{M}^{(n)}] \equiv \mathbf{N}^{(n;2k+1)} =$$

$$= \frac{(-1)^k n}{n+1}\int_{\mathcal{D}} \xi^{2k}\boldsymbol{\xi}^{n-2k-1}\times(\boldsymbol{\xi}\times\boldsymbol{j})\mathrm{d}^3\xi = (-1)^k \mathbf{N}^{((n,1);k)},\quad k=,1,\ldots,$$

where $\mathbf{T}^{(n\ldots);k}$ is the tensor obtained from the contraction of k pairs of indices. Particularly, the tensor $\mathbf{N}^{(2)}$ introduced by equations (14) is represented by the components $\mathbf{N}_i = \mathbf{N}^{(2,1)}_i$.

For the reduction of a totally symmetric tensor to a traceless one we write as in [3, 4]:

$$\tilde{S}_{i_1\ldots i_n} = S_{i_1\ldots i_n} - \sum_{D(i)} \delta_{i_1 i_2} \Lambda[S^{(n)}]_{i_3, i_4 \ldots i_n},$$

where S is a totally symmetric tensor and $\Lambda[S^{(n)}]$ is a totally symmetric tensor of rank $n-2$. Further, we introduce the following notation:

$$\Lambda[\mathbf{M}^{(n)}] = \Lambda^{(n-2)}, \quad \Lambda[\mathbf{P}^{(n)}] = \Pi^{(n-2)}.$$

Moreover, sometimes, for writing some equations in a simpler form, we will employ the notation $\Lambda[\mathbf{T}^{(n)}]$ for $\mathbf{T}^{(n)}$ an arbitrary tensor. By this notation we imply the symmetrisation of $\mathbf{T}^{(n)}$, *i.e.*

$$\Lambda[\mathbf{T}^{(n)}] \equiv \Lambda[\mathbf{T}^{(n)}_{sym}].$$

In [13] a general procedure of detracing a symmetric tensor is presented. In our case the result of this procedure may be written as:

$$\Lambda[S^{(n)}]_{i_3\ldots i_n} = \sum_{m=1}^{[n/2]} \frac{(-1)^{m-1}(2n-1-2m)!!}{(2n-1)!!m} \sum_{D(i)} \delta_{i_3 i_4} \ldots \delta_{i_{2m-1} i_{2m}} S^{n;m}_{i_{2m+1}\ldots i_n}.$$

Using the definitions and notation presented above, it is easy to obtain the following useful results:

a) for detracing electric moment tensors:

$$\Pi = \Lambda[\mathbf{P}^{(2)}]: \quad \Pi = \frac{1}{3}\mathbf{P}_{ii}; \quad \Pi^{(1)} = \Lambda[\mathbf{P}^{(3)}]: \quad \Pi_i = \frac{1}{5}\mathbf{P}_{qqi};$$

$$\Pi^{(2)} = \Lambda[\mathbf{P}^{(4)}]: \quad \Pi_{ij} = \frac{1}{7}\mathbf{P}_{qqij} - \frac{1}{70}\mathbf{P}_{qqll}\delta_{ij};$$

$$\Pi^{(3)} = \Lambda[\mathbf{P}^{(5)}]: \quad \Pi_{ijk} = \frac{1}{11}\mathbf{P}_{qqijk} - \frac{1}{11\times 18}\sum_{D(i)} \delta_{ij}\mathbf{P}_{qqllk};$$

b) for detracing magnetic moments tensors:

$$\Lambda = \Lambda[\mathbf{M}^{(2)}_{sym}] = 0; \ \Lambda^{(1)} = \Lambda[\mathbf{M}^{(3)}_{sym}]: \ \Lambda_i = \frac{1}{15}\mathbf{M}_{qqi};$$

$$\Lambda^{(2)} = \Lambda[\mathbf{M}^{(4)}_{sym}]: \ \Lambda_{ij} = \frac{1}{28}(\mathbf{M}_{qqij} + \mathbf{M}_{qqji});$$

$$\Lambda^{(3)} = \Lambda[\mathbf{M}^{(5)}_{sym}]: \ \Lambda_{ijk} = \frac{1}{45}\sum_{D}\mathbf{M}_{qqijk} - \frac{1}{14\times 45}\sum_{D}\delta_{ij}\mathbf{M}_{qqllk}.$$

5. TOTAL RADIATION POWER IN TERMS OF REDUCED MOMENTS

The usual procedure is to emphasize, in the expression of $\mathcal{J}$, the reduced tensors by utilizing the reduction relations given in the previous section. The well-known notation for dipole moments is used:

$$\mathsf{P}_i = p_i : \boldsymbol{p}, \quad \mathsf{M}_i = m_i : \boldsymbol{m}.$$

The static expressions of the reduced tensors of electric and magnetic polarizations are $\mathcal{P}^{(n)}$ and $\mathcal{M}^{(n)}$ with:

$$\mathcal{P}_{i_1\ldots i_n} = \frac{(-1)^n}{(2n-1)!!}\int_{\mathcal{D}} \rho(\boldsymbol{r},t) r^{2n+1}\boldsymbol{\nabla}^n \frac{1}{r}\mathrm{d}^3x,$$

and as in [3],

$$\mathcal{M}_{i_1\ldots i_n} = \frac{(-1)^n}{(n+1)(2n-1)!!}\sum_{\lambda=1}^{n}\int_{\mathcal{D}} r^{2n+1}[\boldsymbol{j}(\boldsymbol{r},t)\times\boldsymbol{\nabla}]_{i_\lambda}\,\partial^{(\lambda)}_{i_1\ldots i_n}\frac{1}{r}\mathrm{d}^3x.$$

Below, there are the results of the reduction of the first terms in the expansion of $\mathcal{J}$, which correspond to the limitation to the order $(d/\lambda)^6$:

$$4\pi\varepsilon_0 c^3\mathcal{J}^{(0,0)} = \frac{2}{3}\ddot{\boldsymbol{p}}^2,$$

$$4\pi\varepsilon_0 c^3(\mathcal{J}^{(0,2)} + \mathcal{I}^{(2,0)}) = -\frac{4}{3c^2}\ddot{p}_i\dddot{\mathsf{T}}_i, \quad \mathsf{T}_i = \frac{1}{4}\mathsf{N}_i - \frac{1}{6}\dot{\Pi}_i,$$

$$4\pi\varepsilon_0 c^3\mathcal{I}^{(1,1)} = \frac{2}{3c^2}\ddot{\boldsymbol{m}}^2 + \frac{1}{20c^2}\dddot{\mathcal{P}}_{ij}\dddot{\mathcal{P}}_{ij},$$

$$4\pi\varepsilon_0 c^3\mathcal{J}^{(2,2)} = \frac{1}{4c^4}\left[\frac{1}{5}\dddot{\mathcal{M}}_{ij}\dddot{\mathcal{M}}_{ij} + \frac{1}{6}\dddot{\mathsf{N}}_i\dddot{\mathsf{N}}_i - \frac{2}{9}\dddot{\mathsf{N}}_i\dddot{\Pi}_i + \frac{2}{27}\ddddot{\Pi}_i\ddddot{\Pi}_i + \frac{8}{945}\ddddot{\mathcal{P}}_{ijk}\ddddot{\mathcal{P}}_{ijk}\right], \quad (15)$$

$$4\pi\varepsilon_0 c^3(\mathcal{J}^{(1,3)} + \mathcal{J}^{(3,1)}) = \frac{1}{3c^4}\left[\frac{4}{15}\ddot{m}_k\ddddot{\mathsf{M}}_{qqk} - \frac{1}{15}\ddddot{\mathsf{N}}^{(3,1)}_{ij}\dddot{\mathcal{P}}_{ij} + \frac{3}{40}\ddddot{\Pi}_{ij}\dddot{\mathcal{P}}_{ij}\right],$$

$$4\pi\varepsilon_0 c^3(\mathcal{J}^{(0,4)} + \mathcal{J}^{(4,0)}) = \frac{1}{12c^4}\left[-\frac{1}{5}\ddot{p}_i\ddddot{\mathsf{N}}^{(4,1)}_{qqi} + \frac{4}{25}\ddot{p}_i\dddot{\Pi}_{jji}\right]. \quad (16)$$

The toroid dipole moment is defined [10–12]:

$$\mathsf{T}_i = \frac{1}{4}\mathsf{N}_i - \frac{1}{6}\dot{\Pi}_i.$$

We also employed the simplified notation $\mathbf{N}_i = \mathbf{N}_i^{(2,1)}$. $\tilde{\Pi}^{(2)}$ is a traceless tensor (that is $\Pi^{(2)}$ "detraced"):

$$\tilde{\Pi}_{ik} = \Pi_{ik} - \frac{1}{3}\Pi_{qq}\delta_{ik} = \frac{1}{7}\mathbf{P}_{qqik} - \frac{1}{21}\mathbf{P}_{qqll}\delta_{ik}.$$

The sum of terms from the equation (15) represents the total radiated power expanded up to the fourth order with respect to d/λ, that is:

$$\sum_{n+m\le 4} \mathcal{J}^{(n,m)}.$$

One can verify the relation

$$\frac{2}{3}\ddot{\boldsymbol{p}}^2 - \frac{4}{3c^2}\ddot{p}_i\ddot{\mathbf{T}}_i + \frac{1}{24c^4}\ddot{\mathbf{N}}_i\ddot{\mathbf{N}}_i - \frac{1}{18c^4}\ddot{\mathbf{N}}_i\dddot{\Pi}_i + \frac{1}{54}\dddot{\Pi}_i\dddot{\Pi}_i = \frac{2}{3}\left(\ddot{\boldsymbol{p}} - \frac{1}{c^2}\ddot{\mathbf{T}}\right)^2.$$

By eliminating the higher order derivatives, we may write:

$$\frac{1}{15}\mathbf{N}_{ik}\mathcal{P}_{ik} - \frac{3}{40}\dot{\Pi}_{ik}\mathcal{P}_{ik} = \frac{1}{15}\tilde{\mathbf{N}}_{ik}\mathcal{P}_{ik} - \frac{3}{40}\dot{\tilde{\Pi}}_{ik}\mathcal{P}_{ik} = \frac{3}{10}\left(\frac{1}{9}\tilde{\mathbf{N}}_{ik} - \frac{1}{4}\dot{\tilde{\Pi}}_{ik}\right)\mathcal{P}_{ik},$$

with the notation $\mathbf{N}_{ik} = \mathbf{N}_{ik}^{(3,1)}$. The so-called quadrupole toroid moment [10–12, 14] is also defined by:

$$\mathbf{T}_{ik} = \frac{1}{9}\tilde{\mathbf{N}}_{ik} - \frac{1}{4}\dot{\tilde{\Pi}}_{ik}.$$

Explicitly, the two toroid moments are written as:

$$\mathbf{T}_i = \frac{1}{6}\int_{\mathcal{D}}[\boldsymbol{\xi}\times(\boldsymbol{\xi}\times\boldsymbol{j})]_i\,\mathrm{d}^3\xi - \frac{1}{30}\dot{\mathbf{P}}_{qqi} =$$
$$= \frac{1}{6}\int_{\mathcal{D}}[(\boldsymbol{\xi}\cdot\boldsymbol{j})\xi_i - \xi^2 j_i]\mathrm{d}^3\xi - \frac{1}{30}\int_{\mathcal{D}}\xi^2\xi_i\dot{\rho}\,\mathrm{d}^3\xi.$$

Since:

$$\int_{\mathcal{D}}\xi^2\xi_i\dot{\rho}\,\mathrm{d}^3\xi = -\int_{\mathcal{D}}\xi^2\xi_i\nabla\cdot\boldsymbol{j}\,\mathrm{d}^3\xi = \int_{\mathcal{D}}\boldsymbol{j}\cdot\nabla(\xi^2\xi_i)\mathrm{d}^3\xi = \int_{\mathcal{D}}[2(\boldsymbol{\xi}\cdot\boldsymbol{j})\xi_i + \xi^2 j_i]\mathrm{d}^3\xi,$$

it results:

$$\mathbf{T}_i = \frac{1}{10}\int_{\mathcal{D}}[(\boldsymbol{\xi}\cdot\boldsymbol{j})\xi_i - 2\xi^2 j_i]\mathrm{d}^3\xi.$$

In a similar way:

$$\mathbf{T}_{ik} = \frac{1}{42}\int_{\mathcal{D}}[4(\boldsymbol{\xi}\cdot\boldsymbol{j})\xi_i\xi_k - 5\xi^2(\xi_i j_k + \xi_k j_i) + 2\xi^2(\boldsymbol{\xi}\cdot\boldsymbol{j})\delta_{ik}]\mathrm{d}^3\xi.$$

After regrouping the terms and considering the above presented relations, one gets:

$$4\pi\varepsilon_0 c^3 \mathcal{J}^{(4)} = \frac{2}{3}\left(\ddot{\boldsymbol{p}} - \frac{1}{c^2}\dddot{\mathbf{T}}\right)^2 + \frac{2}{3c^2}\ddot{\boldsymbol{m}}^2 + \frac{1}{20c^2}\dddot{\mathcal{P}}_{ij}\dddot{\mathcal{P}}_{ij} - \frac{1}{10c^4}\ddddot{\mathbf{T}}_{ij}\dddot{\mathcal{P}}_{ij} + \\ + \frac{4}{45c^4}\ddot{m}_k\ddddot{\mathbf{M}}_{qqk} - \frac{1}{60c^4}\ddot{p}_i\left(\overset{\cdot\cdot\cdot\cdot\cdot}{\mathbf{N}}{}^{(4,1)}_{qqi} - \frac{4}{5}\overset{\cdot\cdot\cdot\cdot\cdot\cdot}{\Pi}_{qqi}\right) + \frac{1}{20c^4}\dddot{\mathcal{M}}_{ij}\dddot{\mathcal{M}}_{ij} + \\ + \frac{2}{945c^4}\ddddot{\mathcal{P}}_{ijk}\ddddot{\mathcal{P}}_{ijk}. \tag{17}$$

If the expansion of the radiated power up to the 4-th order with respect to d/λ is considered, only the following sum will be kept:

$$\mathcal{J}^{(2)} = \mathcal{J}^{(0,0)} + \mathcal{J}^{(1,1)} + 2\mathcal{I}^{(0,2)} = \\ = \frac{1}{4\pi\varepsilon_0 c^3}\left[\frac{2}{3}\ddot{\boldsymbol{p}}^2 + \frac{2}{3c^2}\ddot{\boldsymbol{m}}^2 - \frac{4}{3c^2}\ddot{p}_i\dddot{\mathbf{T}}_i + \frac{1}{20c^2}\dddot{\mathcal{P}}_{ij}\dddot{\mathcal{P}}_{ij}\right].$$

This is, actually, a result obtained by Belloti and Bornatici [8]. They observe that, compared to the expression for the *dipolar magnetic-quadrupolar electric* radiation from the books of Jackson and Landau [1, 2], and from many other electrodynamics books, this result contains a supplementary term given by the contribution of the vector $\boldsymbol{T}$[1]. We point out that in [1, 2] the goal is only to calculate the isolated contributions of the electric dipole, electric quadrupole and magnetic dipole to the total radiated power without regarding it as a result of an expansion. If one associates a multipole moment to an elementary system, one obtains an isolated contribution of this multipole, but when one considers a composite system, all multipoles, giving contributions of the same order of magnitude, must be considered. This is the reason for which the result from [1, 2] should not be considered erroneous. On the other hand, Bellotti and Bornatici give a correct result, if one considers how far they go with the expansion of the radiated power.

It seems that, in general, in literature, similar results contain a series of confusions due to the inconsistent application of the criteria according to which different terms of this expansion must be compared. For instance, in [12] the following result is presented:

$$4\pi\varepsilon_0 c^3 \mathcal{J}_{Dubovik} = \frac{2}{3}\left(\ddot{\boldsymbol{p}} - \frac{1}{c^2}\dddot{\mathbf{T}}\right)^2 + \frac{2}{3c^2}\ddot{\boldsymbol{m}}^2 + \frac{1}{20c^2}\left(\dddot{\mathcal{P}}_{ik} - \frac{1}{c^2}\dddot{\mathbf{T}}_{ik}\right)\left(\dddot{\mathcal{P}}_{ik} - \frac{1}{c^2}\dddot{\mathbf{T}}_{ik}\right) + \\ + \frac{2}{945c^4}\left(\ddddot{\mathcal{P}}_{ijk}\ddddot{\mathcal{P}}_{ijk} + \ddddot{\mathcal{M}}_{ijk}\ddddot{\mathcal{M}}_{ijk}\right) + \ldots. \tag{18}$$

[1] This result was obtained also by Dubovik *et al.* [10].

Apparently, the radiated power is expanded in (18) up to the 6-th order in d/λ, but comparing this expansion with equation (17), we observe some terms missing in (18) as, for example, $\dddot{\mathcal{M}}_{ij}\dddot{\mathcal{M}}_{ij}$.

In some more recent papers, as for example in [9], it is claimed that, as an application of more general formulas, the expansion up to the order $1/c^5$ is given (that is, exactly how it is considered in [8]). The $1/c$ criterion is correlated with the wavelength criterion if one takes into account the powers of the parameter $1/c$ together with the orders of the partial temporal derivatives in the expansion of the radiated power. The expression given in [9] (written with our notation) is:

$$4\pi\varepsilon_0 c^3 \mathcal{J}_{R-V} = \frac{2}{3}\left(\ddot{\boldsymbol{p}} - \frac{1}{c^2}\dddot{\mathbf{T}}\right)^2 + \frac{4}{45c^4}\ddot{m}_k \ddddot{\mathbf{M}}_{qqk} + \frac{1}{20c^2}\dddot{\mathcal{P}}_{ij}\dddot{\mathcal{P}}_{ij} + \\ + \frac{1}{20c^4}\dddot{\mathcal{M}}_{ij}\dddot{\mathcal{M}}_{ij}. \tag{19}$$

In order to identify it with the equation (4.12) from [9], one must consider the definitions of the quadrupol moments and the identity:

$$\mathbf{M}_{qqk} = \frac{3}{4}\int \xi^2 (\boldsymbol{\xi} \times \boldsymbol{j})_k \,\mathrm{d}^3\xi = \frac{1}{2}\bar{\bar{\rho}}^2.$$

This last parameter is the one used in the equation (19). It is easy to observe that if one tries to interpret equation (19) as the expansion of $\mathcal{J}$ up to the sixth order, many terms are missing.

6. REDUCTION OF THE MULTIPOLE TENSORS AND GAUGE INVARIANCE

In this section, the results from [5, 6] will be presented in a more systematic and concise way. A different explanation of these results is given in [7] using the expansions of ρ and $\boldsymbol{j}$. These results present the possibility of expressing the electromagnetic potentials, as well as the field, exclusively in terms of reduced moments, *i.e.* moments represented by reduced tensors (symmetric and traceless). This procedure allows us to express the radiated power in a very simple general form, as it will be shown below.

Let us consider the expansion of the potentials with the moments $\mathbf{M}^{(n)}$ (see equation (12)) and $\mathbf{P}^{(n)}$ (see equation (2)):

$$\boldsymbol{A}(\boldsymbol{r},t) = \frac{\mu_0}{4\pi}\nabla \times \sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{n!}\nabla^{n-1} \,\|\, \left[\frac{1}{r}\mathbf{M}^{(n)}(t - r/c)\right] +$$

$$+\frac{\mu_0}{4\pi}\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{n!}\nabla^{n-1}\,\|\left[\frac{1}{r}\dot{\mathbf{P}}^{(n)}(t-r/c)\right], \tag{20}$$

$$\Phi(\boldsymbol{r},t)=\frac{1}{4\pi\varepsilon_0}\sum_{n\geq 0}\frac{(-1)^n}{n!}\nabla^n\,\|\left[\frac{\mathbf{P}^{(n)}(t-r/c)}{r}\right]. \tag{21}$$

The basic idea is that the tensors $\mathbf{M}^{(n)}$ and $\mathbf{P}^{(n)}$ can be replaced by reduced tensors. This leads to a modification of some of the inferior order moments, such that finally all the moments up to (and including) the n-th order moment must be expressed by reduced tensors. Not all of them, though, reduce to the static expressions $\mathcal{M}$ and $\mathcal{P}$. The final results for these moments are $\widetilde{\mathbf{M}}$ and $\widetilde{\mathbf{P}}$. Formally, as a result of this procedure, the expressions for the potentials expansions up to the n-th order will be given by the equations (20) and (21) with the substitutions $\mathbf{P}\to\widetilde{\mathbf{P}}$ and $\mathbf{M}\to\widetilde{\mathbf{M}}$.

As a basis for this procedure one could take the following observations resuming systematically the results from [5]:

- Let $\mathbf{L}^{(n)}$ be a $\mathbf{M}^{(n)}$-type tensor, *i.e.* $\mathbf{L}$ is fully symmetric in the first $n-1$ indices, $\mathbf{L}_{i_1\ldots i_{n-1}i_k}=0$ if $k\leq n-1$. The symmetrisation of $\mathbf{L}$ is realized with the relation:

$$\mathbf{L}_{(sym)i_1\ldots i_n}=\mathbf{L}_{i_1\ldots i_n}-\frac{1}{n}\sum_{\lambda=1}^{n-1}\varepsilon_{i_\lambda i_n q}\mathcal{N}^{(\lambda)}_{i_1\ldots i_{n-1}q}[\mathbf{L}^{(n)}],$$

with:

$$\mathcal{N}_{i_1\ldots i_{n-1}}[\mathbf{L}^{(n)}]=\varepsilon_{i_{n-1}ps}\mathbf{L}_{i_1\ldots i_{n-2}ps}.$$

In this case:

a) the substitution:

$$\mathbf{M}_{i_1\ldots i_n}\longrightarrow\mathbf{M}_{(L)i_1\ldots i_n}=\mathbf{M}_{i_1\ldots i_n}-\frac{1}{n}\sum_{\lambda=1}^{n-1}\varepsilon_{i_\lambda i_n q}\mathcal{N}^{(\lambda)}_{i_1\ldots i_{n-1}q}[\mathbf{L}^{(n)}] \tag{22}$$

produces changes of the potentials which, up to a gauge transformation, are compensated by the following transformation:

$$\mathbf{P}^{(n-1)}\longrightarrow\mathbf{P}^{(n-1)}-\frac{n-1}{c^2n^2}\dot{\mathcal{N}}[\mathbf{L}^{(n)}];\qquad (I) \tag{23}$$

b) the substitution:

$$\mathbf{P}_{i_1\ldots i_n}\longrightarrow\mathbf{P}_{(L)i_1\ldots i_n}-\frac{1}{n}\sum_{\lambda=1}^{n-1}\varepsilon_{i_\lambda i_n q}\mathcal{N}^{(\lambda)}_{i_1\ldots i_{n-1}q}[\mathbf{L}^{(n)}]$$

produces changes of the potentials which are compensated by the following transformation:

$$\mathbf{M}^{(n-1)} \longrightarrow \mathbf{M}^{(n-1)} + \frac{n-1}{n^2}\dot{\mathcal{N}}[\mathbf{L}^{(n)}]. \qquad (II) \tag{24}$$

- Let $\mathbf{S}^{(n)}$ be a tensor and the detracing operation for it:

$$\widetilde{\mathbf{S}}_{i_1\ldots i_n} = \mathbf{S}_{i_1\ldots i_n} - \sum_{D(i)} \delta_{i_1 i_2} \Lambda_{i_3\ldots i_n}[\mathbf{S}^{(n)}]. \tag{25}$$

Then:

c) the substitution:

$$\mathbf{M}_{i_1\ldots i_n} \longrightarrow \mathbf{M}_{(S)i_1\ldots i_n} - \sum_{D(i)} \delta_{i_1 i_2} \Lambda_{i_3\ldots i_n}[\mathbf{S}^{(n)}] \tag{26}$$

produces changes of the potentials which, up to a gauge transformation, are compensated by the following transformation:

$$\mathbf{M}^{(n-2)} \longrightarrow \mathbf{M}^{(n-2)} + \frac{n-2}{2c^2 n}\ddot{\Lambda}[\mathbf{S}^{(n)}]; \qquad (III) \tag{27}$$

d) the substitution:

$$\mathbf{P}_{i_1\ldots i_n} \longrightarrow \mathbf{P}_{(S)i_1\ldots i_n} - \sum_{D(i)} \delta_{i_1 i_2} \Lambda_{i_3\ldots i_n}[\mathbf{S}^{(n)}] \tag{28}$$

produces changes of the potentials which, up to a gauge transformation, are compensated by the following transformation:

$$\mathbf{P}^{(n-2)} \longrightarrow \mathbf{P}^{(n-2)} + \frac{n-2}{2c^2 n}\ddot{\Lambda}[\mathbf{S}^{(n)}]. \qquad (IV) \tag{29}$$

These four transformation relations of the electromagnetic potentials are sufficient for the development of a scheme in which the replacement of the multipole moments tensors by symmetric and traceless tensors is presented.

The results of such a scheme, valid for expansions up to the fourth order with respect to d/λ, are given in Tables 1 and 2. It can be easily continued for higher orders. It is possible also to express results for the reduced tensors $\widetilde{\mathbf{P}}^{(n)}$ and $\widetilde{\mathbf{M}}^{(n)}$ for arbitrary n [15].

7. RADIATED POWER EXPRESSED BY REDUCED MOMENTS

By applying the transformations (13) to the expansion of the vector potential of the radiated field, one gets:

$$\boldsymbol{A}_{rad}(\boldsymbol{r}, t) = \frac{\mu_0}{4\pi}\frac{1}{r}\sum_{n=1}^{\infty}\frac{1}{n!c^n}\left[[\boldsymbol{\nu}^{(n-1)} \,\|\, \mathbf{M}^{(n)}_{,n}(t_0)] \times \boldsymbol{\nu} + c\sum_{n=1}^{\infty}[\boldsymbol{\nu}^{(n-1)} \,\|\, \mathbf{P}^{(n)}_{,n}(t_0)]\right].$$

The radiated power (angular distribution) will be given by:

$$
\begin{aligned}
4\pi\varepsilon_0 \mathcal{J}(\boldsymbol{\nu}) = & \sum_{n=1}^{\infty}\sum_{m=1}^{\infty} \frac{1}{n!m!c^{n+m}} \Big[\left(\boldsymbol{\nu}^{(n-1)} \,\|\, \mathbf{M}^{(n)}_{,n+1}\right)\left(\boldsymbol{\nu}^{(m-1)} \,\|\, \mathbf{M}^{(n)}_{,m+1}\right) - \\
& -\left(\boldsymbol{\nu}^{(n)} \,\|\, \mathbf{M}^{(n)}_{,n+1}\right)\left(\boldsymbol{\nu}^{(m)} \,\|\, \mathbf{M}^{(n)}_{,m+1}\right)\Big] + \\
& +\sum_{n=1}^{\infty}\sum_{m=1}^{\infty} \frac{c^2}{n!m!c^{n+m}} \Big[\left(\boldsymbol{\nu}^{(n-1)} \,\|\, \mathbf{P}^{(n)}_{,n+1}\right)\left(\boldsymbol{\nu}^{(m-1)} \,\|\, \mathbf{P}^{(m)}_{,m+1}\right) - \\
& -\left(\boldsymbol{\nu}^{(n)} \,\|\, \mathbf{P}^{(n)}_{,n+1}\right)\left(\boldsymbol{\nu}^{(m)} \mathbf{P}^{(m)}_{,m+1}\right)\Big] + \\
& +\sum_{n=1}^{\infty}\sum_{m=1}^{\infty} \frac{c}{n!m!c^{n+m}} \Big\{ \left(\boldsymbol{\nu}^{(n-1)} \,\|\, \mathbf{M}^{(n)}_{,n+1}\right)\cdot\Big[\boldsymbol{\nu}\times\left(\boldsymbol{\nu}^{(m-1)} \,\|\, \mathbf{P}^{(m)}_{,m+1}\right)\Big] + \\
& +\left(\boldsymbol{\nu}^{(n-1)} \,\|\, \mathbf{M}^{(n)}_{,n+1}\right)\cdot\Big[\boldsymbol{\nu}\times\left(\boldsymbol{\nu}^{(m-1)} \,\|\, \mathbf{P}^{(m)}_{,m+1}\right)\Big]\Big\}.
\end{aligned}
$$

We consider the procedure for the reduction of the moments tensors from the expansion of the vector potential applied up to the μ-th order for the magnetic, and to the ε-th order for the electric moments. The sum of the terms from the expansion of the radiated power which contain *exclusively* magnetic moments reduced up to μ and electric moments up to ε is:

$$
\begin{aligned}
4\pi\varepsilon_0 \mathcal{J}_{\mu,\varepsilon}(\boldsymbol{\nu}) = & \sum_{n=1}^{\infty}\sum_{m=1}^{\infty} \frac{1}{n!m!c^{n+m}} \Big[\left(\boldsymbol{\nu}^{(n-1)} \,\|\, \widetilde{\mathbf{M}}^{(n)}_{,n+1}\right)\left(\boldsymbol{\nu}^{(m-1)} \,\|\, \widetilde{\mathbf{M}}^{(m)}_{,m+1}\right) - \\
& -\left(\boldsymbol{\nu}^{(n)} \,\|\, \widetilde{\mathbf{M}}^{(n)}_{,n+1}\right)\left(\boldsymbol{\nu}^{(m)} \widetilde{\mathbf{M}}^{(m)}_{,m+1}\right)\Big] + \\
& +\sum_{n=1}^{\infty}\sum_{m=1}^{\infty} \frac{c^2}{n!m!c^{n+m}} \Big[\left(\boldsymbol{\nu}^{(n-1)} \,\|\, \widetilde{\mathbf{P}}^{(n)}_{,n+1}\right)\left(\boldsymbol{\nu}^{(m-1)} \,\|\, \widetilde{\mathbf{P}}^{(m)}_{,m+1}\right) - \\
& -\left(\boldsymbol{\nu}^{(n)} \,\|\, \widetilde{\mathbf{P}}^{(n)}_{,n+1}\right)\left(\boldsymbol{\nu}^{(m)} \widetilde{\mathbf{P}}^{(m)}_{,m+1}\right)\Big] + \\
& +\sum_{n=1}^{\infty}\sum_{m=1}^{\infty} \frac{c}{n!m!c^{n+m}} \Big\{ \left(\boldsymbol{\nu}^{(n-1)} \,\|\, \widetilde{\mathbf{M}}^{(n)}_{,n+1}\right)\cdot\Big[\boldsymbol{\nu}\times\left(\boldsymbol{\nu}^{(m-1)} \,\|\, \widetilde{\mathbf{P}}^{(m)}_{,m+1}\right)\Big] + \\
& +\left(\boldsymbol{\nu}^{(n-1)} \,\|\, \widetilde{\mathbf{M}}^{(n)}_{,n+1}\right)\cdot\Big[\boldsymbol{\nu}\times\left(\boldsymbol{\nu}^{(m-1)} \,\|\, \widetilde{\mathbf{P}}^{(m)}_{,m+1}\right)\Big]\Big\}.
\end{aligned}
\tag{30}
$$

It is easy to understand that the above sum cannot be identified with the expansion of the radiated power containing the magnetic moments $\mathbf{M}^{(k)}$, $k \le \mu$ and the electric moments $\mathbf{P}^{(l)}$, $l \le \varepsilon$. This happens because some of the magnetic moments $\widetilde{\mathbf{M}}^{(k)}$ contain tensorial expressions built with magnetic and electric tensors of superior orders in k and similar for the electric case. But, once $\mathcal{J}_{\mu,\varepsilon}$ is settled, the superior orders contributions can be eliminated in order to obtain the correct expansion with the d/λ criterion.

Returning to the expression of the total radiated power as a function of the magnetic moments $\boldsymbol{\mu}^{(n)}$, it results that the expansion (10) corresponds to the expansion (30) for $\mu = M$ and $\varepsilon = M + 1$ when only the contributions of order not higher than M are retained.

According to the results from [6], expressing the total radiated power is easier if one uses the averaging formula (7) and the symmetric and traceless character of the reduced tensors. Hence, the following properties are valid:

a) Let two symmetric traceless tensors $\mathbf{A}^{(n)}$ and $\mathbf{B}^{(n)}$ and their averaged contraction:

$$<\left(\mathbf{v}^k \parallel \mathbf{A}^{(n)}\right) || \left(\mathbf{v}^{k'} \parallel \mathbf{B}^{(m)}\right)> = <\nu_{i_1}\dots\nu_{i_k}\nu_{j_1}\dots\nu_{j'_k}\mathbf{A}_{i_1\dots i_k i_{k+1}\dots i_n}\mathbf{B}_{j_1\dots j'_k j_{k'+1}\dots j_m}> .$$

This average is different from zero only for products $\delta_{i_p j_k}$ with $p = 1, \dots, k$ and $q = 1, \dots, k'$. The following relation is valid:

$$<\left(\mathbf{v}^k \parallel \mathbf{A}^{(n)}\right) || \left(\mathbf{v}^{k'} \parallel \mathbf{B}^{(m)}\right)> = \frac{k!}{(2k+1)!!}\left[\mathbf{A}^{(n)} || \mathbf{B}^{(m)}\right]\delta_{k'k} .$$

b) The terms of the last sums containing mixed moments products in equation (32), give contributions of the type:

$$<\nu_{i_1}\dots\nu_{i_{n-1}}\nu_{j_1}\dots\nu_{j_{m-1}}\nu_p>\varepsilon_{i_n pq}\mathbf{A}_{i_1\dots i_n}\mathbf{B}_{j_1\dots j_{m-1}q} .$$

All the terms from the δ products, representing the averages of the $\mathbf{v}$ components products, contain either $\delta_{i_k p}$ or δ_{pj_l}, $k = 1, \dots, n-1$, $l = 1, \dots, m-1$ such that, because of $\varepsilon_{i_n pq}$ and of the symmetry of A and B, the result is zero. Using these properties one can write:

$$\mathcal{J}_{\mu\varepsilon} = \frac{1}{4\pi\varepsilon_0 c^3}\left[\sum_{n=1}^{\mu}\frac{n+1}{nn!(2n+1)!!c^{2n}}\left(\widetilde{\mathbf{M}}^{(n)}_{,n+1} \parallel \widetilde{\mathbf{M}}^{(n)}_{,n+1}\right) + \right.$$
$$\left. + \sum_{n=1}^{\varepsilon}\frac{n+1}{nn!(2n+1)!!c^{2n-2}}\left(\widetilde{\mathbf{P}}^{(n)}_{,n+1} \parallel \widetilde{\mathbf{P}}^{(n)}_{,n+1}\right)\right].$$

Let us consider now different expressions of $\mathcal{J}^{(M)}$, calculated this time starting from the formula given by the equation written above. As settled, for a given value of M one must consider this formula for $\mu = M$, $\varepsilon = M + 1$.

$\mathcal{J}^{(0)}$: $\mu = 0$, $\varepsilon = 1$, only the electric dipole moment has a contribution:

$$4\pi\varepsilon_0 c^3 \mathcal{J}^{(0)} = \frac{2}{3}\ddot{\boldsymbol{p}}^2 .$$

$\mathcal{J}^{(2)}$: $(\mu, \varepsilon) = (2, 3)$; in this case the results from the Tables 1 and 2 are used, considering in the end only the terms corresponding to the order M:

$$4\pi\varepsilon_0 c^3 \mathcal{J}^{(2)} = \left\{ \frac{2}{3c^2}\ddot{\tilde{\mathbf{M}}}_I \ddot{\tilde{\mathbf{M}}}_I + \frac{1}{20c^4}\dddot{\tilde{\mathbf{M}}}_{ij}\dddot{\tilde{\mathbf{M}}}_{ij} + \frac{2}{3}\ddot{\tilde{\mathbf{P}}}_i \ddot{\tilde{\mathbf{P}}}_i + \frac{1}{20c^2}\dddot{\tilde{\mathbf{P}}}_i \dddot{\tilde{\mathbf{P}}}_i \right\}_2 .$$

The index of the bracket is the maximum order in d/λ which has to be retained. It follows that:

$$4\pi\varepsilon_0 c^3 \mathcal{J}^{(2)} = \left[\frac{2}{3}\ddot{\boldsymbol{p}}^2 + \frac{2}{3c^2}\ddot{\boldsymbol{m}}^2 - \frac{4}{3c^2}\ddot{\boldsymbol{p}}\cdot\dddot{\boldsymbol{T}} + \frac{1}{20c^2}\mathcal{P}_{ij}\mathcal{P}_{ij} \right].$$

$\mathcal{J}^{(4)}$: $(\mu, \varepsilon) = (4, 5)$

$$\begin{aligned}
4\pi\varepsilon_0 c^3 J^{(4)} = \Bigg\{ & \frac{2}{3c^2}\Big(\ddot{\mathbf{M}} + \frac{1}{6c^2}\ddddot{\boldsymbol{\Lambda}} - \frac{1}{18c^2}\dddot{\mathbf{N}}^{(3,2)}\Big)_i \\
& + \frac{1}{20c^4}\left(\dddot{\mathcal{M}} + \frac{1}{4c^2}\overset{\cdot\cdot\cdot\cdot\cdot}{\tilde{\boldsymbol{\Lambda}}} - \frac{1}{24c^2}\overset{\cdot\cdot\cdot\cdot\cdot}{\tilde{\mathbf{N}}}^{(4,2)} \right)_{ij} \left(\dddot{\mathcal{M}} + \frac{1}{4c^2}\overset{\cdot\cdot\cdot\cdot\cdot}{\tilde{\boldsymbol{\Lambda}}} - \frac{1}{24c^2}\overset{\cdot\cdot\cdot\cdot\cdot}{\tilde{\mathbf{N}}}^{(4,2)} \right)_{ij} \\
& + \frac{2}{945c^6}\ddddot{\mathcal{M}}_{ijk}\ddddot{\mathcal{M}}_{ijk} + \frac{1}{18144c^8}\overset{\cdot\cdot\cdot\cdot\cdot}{\mathcal{M}}_{ijkl}\overset{\cdot\cdot\cdot\cdot\cdot}{\mathcal{M}}_{ijkl} + \\
& + \frac{2}{3}\left(\ddot{\mathbf{P}} - \frac{1}{c^2}\dddot{\mathbf{T}} - \frac{1}{32c^4}\ddddot{\boldsymbol{\Lambda}}[\mathbf{N}^{(4,1)}_{sym}] + \frac{1}{20c^4}\overset{\cdot\cdot\cdot\cdot\cdot\cdot}{\boldsymbol{\Lambda}}[\Pi^{(3)}] + \frac{1}{96c^4}\overset{\cdot\cdot\cdot\cdot\cdot}{\mathbf{N}}^{(4,3)} \right)_i \\
& \left(\ddot{\mathbf{P}} - \frac{1}{c^2}\dddot{\mathbf{T}} - \frac{1}{32c^4}\ddddot{\boldsymbol{\Lambda}}[\mathbf{N}^{(4,1)}_{sym}] + \frac{1}{20c^4}\overset{\cdot\cdot\cdot\cdot\cdot\cdot}{\boldsymbol{\Lambda}}[\Pi^{(3)}] + \frac{1}{96c^4}\overset{\cdot\cdot\cdot\cdot\cdot}{\mathbf{N}}^{(4,3)} \right)_i \\
& + \frac{1}{20c^2}\Big(\dddot{\mathcal{P}} - \frac{1}{c^2}\ddddot{\mathbf{T}}\Big)_{ij}\Big(\dddot{\mathcal{P}} - \frac{1}{c^2}\ddddot{\mathbf{T}}\Big)_{ij} + \frac{2}{945c^4}\Big(\ddddot{\mathcal{P}} - \frac{1}{c^2}\overset{\cdot\cdot\cdot\cdot\cdot}{\mathbf{T}}\Big)_{ijk}\Big(\ddddot{\mathcal{P}} - \frac{1}{c^2}\overset{\cdot\cdot\cdot\cdot\cdot}{\mathbf{T}}\Big)_{ijk} + \\
& + \frac{1}{2^5 \times 3^4 \times 7}\overset{\cdot\cdot\cdot\cdot\cdot}{\mathcal{P}}_{ijkl}\overset{\cdot\cdot\cdot\cdot\cdot}{\mathcal{P}}_{ijkl} + \frac{1}{4 \times 3^3 \times 5^3 \times 77}\overset{\cdot\cdot\cdot\cdot\cdot\cdot}{\mathcal{P}}_{ijklq}\overset{\cdot\cdot\cdot\cdot\cdot\cdot}{\mathcal{P}}_{ijklq} \Bigg\}_4 .
\end{aligned}$$

Only the fourth order terms must be kept from this expansion. This could be done by eliminating the useless terms, but, for obvious reasons, a detailed description is prefered here. We consider Tables 1 and 2 containing the results of reductions:

In the above formulas we introduced the electric toroid moment of undefined order, as it results from the algorithm presented before:

$$\mathbf{T}^{(n)} = \frac{n}{(n+1)^2}\tilde{\mathbf{N}}^{(n+1,1)} - \frac{n}{2(n+2)}\dot{\tilde{\Pi}}^{(n)} .$$

Table 1

	$(\mu, \varepsilon) = (1, 2)$	$(\mu, \varepsilon) = (2, 3)$	$(\mu, \varepsilon) = (3, 4)$
$\tilde{\mathbf{P}}^{(1)}$	$\mathsf{P}^{(1)} : (\boldsymbol{p})$	$\mathbf{P}^{(1)} - \frac{1}{c^2}\dot{\mathbf{T}}^{(1)}$	$\mathbf{P}^{(1)} - \frac{1}{c^2}\dot{\mathbf{T}}^{(1)}$
$\tilde{\mathbf{P}}^{(2)}$	$\mathcal{P}^{(2)} : \mathcal{P}_{ij} = \mathbf{P}_{ij} - \frac{1}{3}\mathbf{P}_{qq}\delta_{ij}$	$\mathcal{P}^{(2)}$	$\mathcal{P}^{(2)} - \frac{1}{c^2}\dot{\mathbf{T}}^{(2)}$
$\tilde{\mathbf{P}}^{(3)}$		$\mathcal{P}^{(3)}$	$\mathcal{P}^{(3)}$
$\tilde{\mathbf{P}}^{(4)}$			$\mathcal{P}^{(4)}$
$\tilde{\mathbf{P}}^{(5)}$			
$\tilde{\mathbf{M}}^{(1)}$	$\mathbf{M}^{(1)} : (\boldsymbol{m})$	$\mathbf{M}^{(1)}$	$\mathbf{M}^{(1)} + \frac{1}{6c^2}\ddot{\Lambda}^{(1)} - \frac{1}{18c^2}\ddot{\mathbf{N}}^{(3,2)}$
$\tilde{\mathbf{M}}^{(2)}$		$\mathcal{M}^{(2)}$	$\mathcal{M}^{(2)}$
$\tilde{\mathbf{M}}^{(3)}$			$\mathcal{M}^{(3)}$
$\tilde{\mathbf{M}}^{(4)}$			

Table 2

	$(\mu, \varepsilon) = (4, 5)$
$\tilde{\mathbf{P}}^{(1)}$	$\mathbf{P}^{(1)} - \frac{1}{c^2}\dot{\mathbf{T}}^{(1)} - \frac{1}{32c^4}\ddot{\Lambda}[\mathbf{N}^{(4,1)}_{sym}] + \frac{1}{20c^4}\dddot{\Lambda}[\Pi^{(3)}] + \frac{1}{96c^4}\dddot{\mathbf{N}}^{(4,3)}$
$\tilde{\mathbf{P}}^{(2)}$	$\mathcal{P}^{(2)} - \frac{1}{c^2}\dot{\mathbf{T}}^{(2)}$
$\tilde{\mathbf{P}}^{(3)}$	$\mathcal{P}^{(3)} - \frac{1}{c^2}\dot{\mathbf{T}}^{(3)}$
$\tilde{\mathbf{P}}^{(4)}$	$\mathcal{P}^{(4)}$
$\tilde{\mathbf{P}}^{(5)}$	$\mathcal{P}^{(5)}$
$\tilde{\mathbf{M}}^{(1)}$	$\mathbf{M}^{(1)} + \frac{1}{6c^2}\ddot{\Lambda}^{(1)} - \frac{1}{18c^2}\ddot{\mathbf{N}}^{(3,2)}$
$\tilde{\mathbf{M}}^{(2)}$	$\mathcal{M}^{(2)} + \frac{1}{4c^2}\ddot{\tilde{\Lambda}}^{(2)} - \frac{1}{24c^2}\ddot{\tilde{\mathbf{N}}}^{(4,2)}$
$\tilde{\mathbf{M}}^{(3)}$	$\mathcal{M}^{(3)}$
$\tilde{\mathbf{M}}^{(4)}$	$\mathcal{M}^{(4)}$

By examining the tables, the terms corresponding to the considered approximation are retained from the expressions representing the squares of these tensors. The result is:

$$4\pi\varepsilon_0 c^3 \mathcal{J}^{(4)} = \frac{2}{3}\left(\ddot{\boldsymbol{p}} - \frac{1}{c^2}\dddot{\mathbf{T}}\right)^2 + \frac{4}{3}\ddot{p}_i\left(-\frac{1}{32c^4}\ddddot{\mathbf{\Lambda}}[\mathbf{N}^{(4,1)}_{sym}] + \frac{1}{20c^4}\ddddot{\mathbf{\Lambda}}[\Pi^{(3)}] + \right.$$

$$+\frac{1}{96c^4}\ddddot{\mathbf{N}}^{(4,3)}\bigg)_i+\frac{1}{20c^2}\dddot{\mathcal{P}}_{ij}\dddot{\mathcal{P}}_{ij}-\frac{1}{10c^4}\dddot{\mathcal{P}}_{ij}\ddddot{\mathbf{T}}_{ij}+\frac{2}{945c^4}\ddddot{\mathcal{P}}_{ijk}\ddddot{\mathcal{P}}_{ijk}+$$

$$+\frac{2}{3c^2}\ddot{\boldsymbol{m}}^2+\frac{4}{3c^2}\ddot{m}_i\left(\frac{1}{6c^2}\ddddot{\boldsymbol{\Lambda}}-\frac{1}{18c^2}\ddddot{\mathbf{N}}^{(3,2)}\right)_i+\frac{1}{20c^4}\dddot{\mathcal{M}}_{ij}\dddot{\mathcal{M}}_{ij}.$$

Because

$$\mathbf{N}^{(4,1)}_{qqi}-\frac{4}{5}\dot{\Pi}_{qqi}=\frac{5}{2}\Lambda_i[\mathbf{N}^{(4,1)}_{sym}]-4\dot{\Lambda}_i[\Pi^{(3)}]-\frac{5}{6}\mathbf{N}^{(4,3)}_i=$$

$$=\frac{2}{7}\int_{\mathcal{D}}[2\xi^2(\boldsymbol{\xi}\cdot\boldsymbol{j})\xi_i-3\xi^4 j_i]\mathrm{d}^3\xi,$$

and

$$\Lambda_i-\frac{1}{3}\mathbf{N}^{(3,2)}_i=\frac{2}{5}\mathbf{M}_{qqi},$$

the last expression of $\mathcal{J}^{(4)}$ is the same as the expression given in equation (17).

8. CONCLUSIONS

The multipole expansions in Cartesian coordinates are not largely treated in literature. Our results were compared to other results from literature, usually obtained by expansion in spherical coordinates. It was underlined the fact that the problem is not accurately treated everywhere.

The main difficulty in the case of Cartesian coordinates is the procedure of reduction of the n-th order multipole tensors to symmetric traceless ones to obtain in this way a description of multipoles in terms of irreducible rotation group representations. The transformations implied in the reduction procedure were defined such that the electromagnetic potentials are altered only by gauge transformations. This implied a specific feature of the dynamic case: the redefinitions of the multipole tensors in the lower $n < N$ orders, induced by the reduction of tensors in a given order N.

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