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SPECIAL SOLUTIONS OF THE BOHR HAMILTONIAN RELATED TO SHAPE PHASE TRANSITIONS IN NUCLEI

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Abstract. Nuclei exhibit quantum phase transitions (earlier called ground state phase transitions) between different shapes as the number of nucleons is modified, resulting in changes in the ground and low lying nuclear states. Special solutions of the Bohr Hamiltonian appropriate for the critical point of such shape phase transitions, as well as other special solutions applicable to relevant nuclear regions are described.

Key words: shape phase transitions, critical point symmetries, Bohr Hamiltonian.

1. INTRODUCTION

Atomic nuclei are known to exhibit changes of their energy levels and electromagnetic transition rates among them when the number of protons and/or neutrons is modified, resulting in shape phase transitions from one kind of collective behaviour to another. These transitions are not phase transitions of the usual thermodynamic type. They are quantum phase transitions [1] (initially called ground state phase transitions [2]), occurring in Hamiltonians of the type $H = c(H_1 + gH_2)$, where c is a scale factor, g is the control parameter, and H_1, H_2 describe two different phases of the system. The expectation value of a suitably chosen operator, characterizing the state of the system, is used as the order parameter.

In the framework of the Interacting Boson Model [3], which describes nuclear structure of even–even nuclei within the U(6) symmetry, possessing the U(5), SU(3), and O(6) limiting dynamical symmetries, appropriate for

vibrational, axially deformed, and γ -unstable nuclei respectively, shape phase transitions have been studied 25 years ago [2] using the classical limit of the model [4, 5, 6], pointing out that there is (in the usual Ehrenfest classification) a second order shape phase transition between U(5) and O(6), a first order shape phase transition between U(5) and SU(3), and no shape phase transition between O(6) and SU(3). It is instructive to place [1] these shape phase transitions on the symmetry triangle of the IBM [7], at the three corners of which the three limiting symmetries of the IBM appear.

More recently it has been realized [8, 9] that the properties of nuclei lying at the critical point of a shape phase transition can be described by appropriate special solutions of the Bohr Hamiltonian [10], labelled as critical point symmetries. The E(5) critical point symmetry [8] has been found to correspond to the second order critical point between U(5) and O(6), while the X(5) critical point symmetry [9] has been found to correspond to the first order transition between U(5) and SU(3).

The introduction of the critical point symmetries E(5) [8] and X(5) [9] has triggered much work on special solutions of the Bohr Hamiltonian, corresponding to different physical situations. Several of these solutions will be mentioned here, together with experimental examples appropriate for each case.

2. E(5) AND RELATED SOLUTIONS

The original Bohr Hamiltonian [10] is

$$H = -\frac{\hbar^2}{2B} \left[\frac{1}{\beta^4} \frac{\partial}{\partial \beta} \beta^4 \frac{\partial}{\partial \beta} + \frac{1}{\beta^2 \sin 3\gamma} \frac{\partial}{\partial \gamma} \sin 3\gamma \frac{\partial}{\partial \gamma} - \frac{1}{4\beta^2} \sum_{k=1,2,3} \frac{Q_k^2}{\sin^2(\gamma - \frac{2}{3}\pi k)} \right] + V(\beta, \gamma), \quad (1)$$

where β and γ are the usual collective coordinates describing the shape of the nuclear surface, Q_k ($k = 1, 2, 3$) are the components of angular momentum, and B is the mass parameter.

It has been known for a long time [11] that exact separation of variables occurs in the corresponding Schrödinger equation if potentials of the form $V(\beta, \gamma) = U(\beta)$ are used, i.e. if the potential is independent of the variable γ , thus corresponding to γ -soft nuclei. Then wavefunctions of the form $\Psi(\beta, \gamma, \theta_i) = f(\beta)\Phi(\gamma, \theta_i)$ are used, where θ_i ($i = 1, 2, 3$) are the Euler angles describing the orientation of the nucleus in space.

In the equation involving the angles, the eigenvalues of the second order Casimir operator of SO(5) occur, having the form $\Lambda = \tau(\tau + 3)$, where $\tau = 0$,

1, 2, ... is the quantum number characterizing the irreducible representations (irreps) of $SO(5)$, called the "seniority" [12]. This equation has been solved by Bès [13].

The values of angular momentum L contained in each irrep of $SO(5)$ (i.e. for each value of τ) are given by the algorithm [3] $\tau = 3\nu_\Delta + \lambda$, where $\nu_\Delta = 0, 1, \dots$ is the missing quantum number in the reduction $SO(5) \supset SO(3)$, and $L = \lambda, \lambda + 1, \dots, 2\lambda - 2, 2\lambda$ (with $2\lambda - 1$ missing). The values of L allowed for each (τ, ν_Δ) have been tabulated in [3, 14, 15].

The "radial" equation can be simplified by introducing [8] reduced energies $\epsilon = \frac{2B}{\hbar^2}E$ and reduced potentials $u = \frac{2B}{\hbar^2}U$. The form of the solution of the radial equation depends on the choice made for $U(\beta)$.

2.1. E(5)

In the case of E(5) [8, 15], a 5-dimensional (5-D) infinite well [$u(\beta) = 0$ if $\beta \leq \beta_W$, $u(\beta) = \infty$ for $\beta > \beta_W$] is used, since the potential is expected to be flat at the point of a second order shape phase transition. Then the β -equation becomes a Bessel equation of order $\nu = \tau + 3/2$, with eigenfunctions proportional to the Bessel functions $J_{\tau+3/2}(z)$ (with $z = \beta k$, $k = \sqrt{\epsilon}$), while the spectrum is determined by the zeros of the Bessel functions

$$E_{\xi,\tau} = \frac{\hbar^2}{2B} k_{\xi,\tau}^2, \quad k_{\xi,\tau} = \frac{x_{\xi,\tau}}{\beta_W}, \quad (2)$$

where $x_{\xi,\tau}$ is the ξ -th zero of the Bessel function $J_{\tau+3/2}(z)$. The spectrum is parameter free, up to an overall scale factor, which is fixed by normalizing the energies to the excitation energy of the first excited 2^+ state, 2_1^+ . The $R_{4/2} = E(4_1^+)/E(2_1^+)$ ratio turns out to be 2.199. The same holds for the $B(E2)$ values, which are normalized to the $B(E2)$ connecting the two lowest states, $B(E2; 2_1^+ \rightarrow 0_1^+)$. The symmetry present in this case is [15] $E(5) \supset SO(5) \supset SO(3) \supset SO(2)$.

2.2. OTHER SOLUTIONS

For $u(\beta) = \beta^2/2$ one obtains the original solution of Bohr [10, 16], which corresponds to a 5-D harmonic oscillator characterized by the symmetry $U(5) \supset SO(5) \supset SO(3) \supset SO(2)$ [17], the eigenfunctions being proportional to Laguerre polynomials [18], and the spectrum having the simple form $E_N = N + 5/2$, with $N = 2\nu + \tau$, and $\nu = 0, 1, 2, 3, \dots$, which has $R_{4/2} = 2$. The spectra of the $u(\beta) = \beta^2/2$ potential and of the E(5) model become directly comparable by establishing the formal correspondence $\nu = \xi - 1$.

The Davidson potential $u(\beta) = \beta^2 + \frac{\beta_0^4}{\beta^2}$ (where β_0 is the position of the minimum of the potential) [19, 20, 21] also leads to eigenfunctions which are Laguerre polynomials, the energy eigenvalues being [20, 21] (in $\hbar\omega = 1$ units)

$$E_{n,\tau} = 2n + 1 + \left[\left(\tau + \frac{3}{2} \right)^2 + \beta_0^4 \right]^{1/2}. \quad (3)$$

For $\beta_0 = 0$ the above mentioned original solution of Bohr [U(5)] is obtained, while for $\beta_0 \rightarrow \infty$ the O(6) limit of the IBM is obtained [20]. Therefore the Davidson potential provides a one-parameter bridge between U(5) and O(6). One can exploit this fact, by introducing a variational procedure [22, 23], in which the rates of change of the $R_L = E(L_1^+)/E(2_1^+)$ energy ratios of the ground state band with respect to the parameter β_0 are maximized for each L separately. The results lead to an energy spectrum very close to that of E(5) [22]. The method has also been applied to other bands, as well as to $B(E2)$ transition rates [23].

The sequence of potentials $u_{2n}(\beta) = \frac{\beta^{2n}}{2}$ (with n being an integer) leads for $n = 1$ to the Bohr case, while for $n \rightarrow \infty$ leads to the infinite well of E(5) [24]. Therefore this sequence of potentials provides a “bridge” between the U(5) symmetry and the E(5) model, using their common $\text{SO}(5) \supset \text{SO}(3)$ chain of subalgebras for the classification of the spectra. Solutions for $n \neq 1$ have been obtained numerically [25, 26, 14]. The solutions for $n = 2, 3, 4$, labelled as E(5)- β^4 , E(5)- β^6 , and E(5)- β^8 , lead to $R_{4/2} = 2.093, 2.135$, and 2.157 respectively. Complete level schemes have been given in Ref. [14].

A bridge complementary to the one just mentioned, i.e. a bridge spanning the region between E(5) and the γ -soft rotor O(5), has been obtained by using an infinite well potential with boundaries $\beta_M > \beta_m > 0$ [27]. The model, called O(5)-CBS, since it is a γ -soft analog of the confined β -soft (CBS) rotor model [28, 29], contains one free parameter, $r_\beta = \beta_m/\beta_M$, the value $r_\beta = 0$ corresponding to the E(5) model, and $r_\beta \rightarrow 1$ giving the γ -soft rotor [O(5)] limit.

Other solutions obtained in this framework are listed below.

- a) A version of E(5) using a well of finite depth, instead of an infinite one, has been developed [30].
- b) The sextic oscillator, which is a quasi-exactly soluble [31, 32] potential, has also been used as a γ -independent potential [33].
- c) Coulomb-like and Kratzer-like potentials have been used in Ref. [34].
- d) A linear potential has been considered in Ref. [35], where a review of potentials used in this framework is given.
- e) A hybrid model employing a harmonic oscillator for $L \leq 2$ and an infinite square well potential for $L \geq 4$ has been developed [36].

2.3. EXPERIMENTAL MANIFESTATIONS OF E(5)

The first nucleus to be identified as exhibiting E(5) behaviour was ^{134}Ba [37], while ^{102}Pd [38] also seems to provide a very good candidate. Further studies on ^{134}Ba [39] and ^{102}Pd [40], in which no backbending occurs in the ground state band, which remains in excellent agreement with the parameter-free E(5) predictions up to high angular momenta, reinforced this conclusion. ^{104}Ru [41], ^{108}Pd [42], ^{114}Cd [43], and ^{130}Xe [44] have also been suggested as possible candidates. A systematic search [45, 46] on available data on energy levels and B(E2) transition rates suggested ^{102}Pd , $^{106,108}\text{Cd}$, ^{124}Te , ^{128}Xe , and ^{134}Ba as possible candidates, singling out ^{128}Xe as the best one, in addition to ^{134}Ba . This is in agreement with a recent report [47] on measurements of E1 and M1 strengths of $^{124-136}\text{Xe}$ carried out at Stuttgart, which provides evidence for a shape phase transition around $A \simeq 130$. ^{128}Xe has been measured (November 2006) in Jyväskylä [48]. Recently, ^{58}Cr has also been suggested as a candidate [49].

The assumption of a flat β -potential in the E(5) symmetry has been tested by constructing potential energy surfaces (PESs) for nuclei close to the E(5) symmetry, through the use of relativistic mean field theory [50]. It has been found that the relevant PESs come out quite flat, corroborating the E(5) assumption.

2.4. ODD NUCLEI: E(5/4) AND E(5/12)

The models discussed so far are appropriate for even–even nuclei. Odd nuclei can be treated by coupling E(5), describing the even–even part of an odd nucleus, to the odd nucleon by the five-dimensional spin–orbit interaction [51, 15]. If the odd nucleon is in a $j = 3/2$ level, the E(5/4) model [51, 15] occurs, while if the odd nucleon lives in a system of levels with $j = 1/2, 3/2, 5/2$, the E(5/12) model [52] is obtained. Shape phase transitions from spherical to γ -unstable shapes in odd nuclei have also been considered [53, 54] in the framework of the Interacting Boson Fermion Model [55, 56] for the case of an odd nucleon in a $j = 3/2$ level. Shape phase transitions in odd nuclei have also been considered [57] for the case of an odd nucleon in a system of levels with $j = 1/2, 3/2, 5/2$ in the framework of the U(5/12) supersymmetry [55, 56], giving good results in the Os–Hg region.

A first effort to locate nuclei exhibiting the E(5/4) symmetry has been carried out for ^{135}Ba [58], with mixed results. The Ir–Au region might be a more appropriate one, since the U(6/4) supersymmetry has been found there [55, 56]. ^{63}Cu , despite its small size, could also be a good candidate for E(5/4), as discussed in Ref. [51].

3. X(5) AND RELATED SOLUTIONS

In the case of X(5) one tries to solve the Bohr Hamiltonian of Eq. (1) for potentials of the form $u(\beta, \gamma) = u(\beta) + u(\gamma)$, seeking solutions of the relevant Schrödinger equation having the form $\Psi(\beta, \gamma, \theta_i) = \phi_K^L(\beta, \gamma) \mathcal{D}_{M,K}^L(\theta_i)$, where θ_i ($i = 1, 2, 3$) are the Euler angles, $\mathcal{D}(\theta_i)$ denote Wigner functions of them, L are the eigenvalues of angular momentum, while M and K are the eigenvalues of the projections of angular momentum on the laboratory-fixed z -axis and the body-fixed z' -axis respectively. One is interested in cases near axial symmetry, i.e. close to $\gamma = 0$. Thus one uses a harmonic oscillator potential $u(\gamma) = (3c)^2 \gamma^2 / 2$. Near $\gamma = 0$ the last term in the Bohr Hamiltonian can be rewritten in the form [9]

$$\sum_{k=1,2,3} \frac{Q_k^2}{\sin^2(\gamma - \frac{2\pi}{3}k)} \approx \frac{4}{3}(Q_1^2 + Q_2^2 + Q_3^2) + Q_3^2 \left(\frac{1}{\sin^2 \gamma} - \frac{4}{3} \right). \quad (4)$$

Using this result in the Schrödinger equation corresponding to the Hamiltonian of Eq. (1), introducing reduced energies $\epsilon = 2BE/\hbar^2$ and reduced potentials $u = 2BV/\hbar^2$ as in the E(5) case, and taking into account that the reduced potential is of the form $u(\beta, \gamma) = u(\beta) + u(\gamma)$, the Schrödinger equation can be separated into two equations [9].

In the equation containing the γ -variable, β^2 denominators remain, which are replaced by their average values over the β wavefunctions, $\langle \beta^2 \rangle$. Taking into account the simplifications imposed by γ being close to zero, the relevant equation takes the form corresponding to a two-dimensional harmonic oscillator in γ , having wavefunctions proportional to Laguerre polynomials [9].

The form of the solution of the radial equation depends on the choice made for $u(\beta)$.

3.1. X(5)

In X(5) [9] a 5-D infinite well potential is used, as in E(5). The relevant equation is again a Bessel equation, but with order

$$\nu = \left(\frac{L(L+1)}{3} + \frac{9}{4} \right)^{1/2}. \quad (5)$$

The solutions still have the form of Eq. (2), with (ξ, τ) replaced by (s, L) , where s is the order of the relevant root of the Bessel function $J_\nu(k_{s,L}\beta)$. The relevant exactly soluble model is labelled as X(5) (which is not meant as a group label, although there is relation to projective representations of E(5)),

the Euclidean group in 5 dimensions [9]). The total energy has the form

$$E(s, L, n_\gamma, K, M) = E_0 + B(x_{s,L})^2 + An_\gamma + CK^2, \quad (6)$$

where n_γ is the quantum number of the two-dimensional oscillator occurring in the γ -equation, while E_0, A, B, C are free parameters. From this equation it is clear that the spectra of the ground state and β bands only depend on an arbitrary scale, fixed by normalizing them to the energy of the 2_1^+ state, while the bandheads of the γ bands are parameter dependent. (The spacings within the γ bands are however fixed [59].) The $R_{4/2}$ ratio is 2.904.

3.2. OTHER SOLUTIONS

For $u(\beta) = \beta^2/2$ one obtains an exactly soluble model which has been called X(5)- β^2 [60], the eigenfunctions being proportional to Laguerre polynomials and the spectrum having the form

$$E_{n,L} = 2n + 1 + \sqrt{\frac{9}{4} + \frac{L(L+1)}{3}}, \quad n = 0, 1, 2, \dots \quad (7)$$

with $R_{4/2} = 2.646$. The spectra of the $u(\beta) = \beta^2/2$ potential and of the X(5) model become directly comparable by establishing the formal correspondence $n = s - 1$, where n is the usual oscillator quantum number.

The Davidson potential $u(\beta) = \beta^2 + \frac{\beta_0^4}{\beta^2}$ (where β_0 is the position of the minimum of the potential) [19, 20, 21] also leads to eigenfunctions which are Laguerre polynomials, the energy eigenvalues being [22, 23] (in $\hbar\omega = 1$ units)

$$E_{n,L} = 2n + 1 + \left[\frac{1}{3}L(L+1) + \frac{9}{4} + \beta_0^4 \right]^{1/2}. \quad (8)$$

For $\beta_0 = 0$ the above mentioned X(5)- β^2 solution is obtained, while for $\beta_0 \rightarrow \infty$ the rigid rotor limit is obtained. Therefore the Davidson potential provides a one-parameter bridge between X(5)- β^2 and the rigid rotor. One can exploit this fact, by introducing a variational procedure [22, 23], in which the rates of change of the $R_L = E(L_1^+)/E(2_1^+)$ energy ratios of the ground state band with respect to the parameter β_0 are maximized for each L separately. The results lead to an energy spectrum very close to that of X(5) [22]. The method has also been applied to other bands, as well as to $B(E2)$ transition rates [23].

The sequence of potentials $u_{2n}(\beta) = \frac{\beta^{2n}}{2}$ (with n being an integer) leads for $n = 1$ to the X(5)- β^2 case, while for $n \rightarrow \infty$ leads to the infinite well of X(5) [24]. Therefore this sequence of potentials provides a "bridge" between the X(5)- β^2 solution and the X(5) model, in the region lying between U(5) and X(5). Solutions for $n \neq 1$ have been obtained numerically [60]. The solutions

for $n = 2, 3, 4$, labelled as $X(5)-\beta^4$, $X(5)-\beta^6$, and $X(5)-\beta^8$, lead to $R_{4/2} = 2.769$, 2.824, and 2.852 respectively. Complete level schemes have been given in Ref. [60].

A bridge complementary to the one just mentioned, i.e. a bridge spanning the region between $X(5)$ and the rigid rotor, has been obtained by using an infinite well potential with boundaries $\beta_M > \beta_m > 0$ [28, 29]. The model, called the confined β -soft (CBS) rotor model [28, 29], contains one free parameter, $r_\beta = \beta_m/\beta_M$, the value $r_\beta = 0$ corresponding to the $X(5)$ model, and $r_\beta \rightarrow 1$ giving the rigid rotor limit.

Other solutions obtained in this framework are listed below.

a) A potential with linear sloped walls has been considered in Ref. [61]. The sloped walls result in a slower increase of the energy levels of the β band as a function of L in this model, as compared to $X(5)$. This feature improves agreement to experiment.

b) Coulomb-like and Kratzer-like potentials have been used in Ref. [62].

c) The approximate separation of variables used in $X(5)$ has been tested recently through exact numerical diagonalization of the Bohr Hamiltonian [63], using a recently introduced [64, 65, 66] computationally tractable version of the Bohr–Mottelson collective model.

d) Exact separation of variables can be achieved by using potentials of the form $u(\beta, \gamma) = u(\beta) + u(\gamma)/\beta^2$ [11]. This possibility has been recently exploited for the construction of exactly separable (ES) analogues of the $X(5)$ and $X(5)-\beta^2$ models, labelled as ES- $X(5)$ and ES- $X(5)-\beta^2$ respectively [67].

e) The use of periodic $u(\gamma)$ potentials in the $X(5)$ framework has been recently considered in Ref. [68].

A review of potentials used in this framework is given in Ref. [35].

3.3. $X(3)$

The special case in which γ is frozen to $\gamma = 0$, while an infinite square well potential is used in β , leads to an exactly separable three-dimensional model, which has been called $X(3)$ [69]. This model involves three variables, β and the two angles used in spherical coordinates, since the condition $\gamma = 0$ guarantees an axially symmetric prolate shape, for which the two angles of the spherical coordinates suffice for determining its orientation in space.

Exact separation of variables is possible in this case. The equation involving the angles has the usual spherical harmonics as eigenfunctions, the relevant eigenvalues being $L(L+1)$, while the β -equation, in which an infinite square well potential is used, takes the form of a Bessel equation. The radial

solutions have the same form as in X(5), but with order

$$\nu = \sqrt{\frac{L(L+1)}{3} + \frac{1}{4}}, \quad (9)$$

which should be compared to Eq. (5). It should be noticed that in E(3), the Euclidean algebra in 3 dimensions, which is the semidirect sum of the T_3 algebra of translations in 3 dimensions and the SO(3) algebra of rotations in 3 dimensions [70], the eigenvalue equation of the square of the total momentum, which is a second-order Casimir operator of the algebra, also leads [70, 14] to a similar solution, but with $\nu = L + \frac{1}{2} = \sqrt{L(L+1) + \frac{1}{4}}$.

From the symmetry of the wave functions with respect to the plane which is orthogonal to the symmetry axis of the nucleus and goes through its center, follows that the angular momentum L can take only even nonnegative values. Therefore no γ -bands appear in the model, as expected, since the γ degree of freedom has been frozen. The $R_{4/2}$ ratio is 2.44. Complete level schemes have been given in [69].

3.4. EXPERIMENTAL MANIFESTATIONS OF X(5)

The first nucleus to be identified as exhibiting X(5) behaviour was ^{152}Sm [71], followed by ^{150}Nd [72]. Further work on ^{152}Sm [73, 74, 75, 59] and ^{150}Nd [74, 75, 76] reinforced this conclusion. The neighbouring N=90 isotones ^{154}Gd [77, 78] and ^{156}Dy [78, 79] were also seen to provide good X(5) examples, the latter being of inferior quality. In the heavier region, ^{162}Yb [80] and ^{166}Hf [81] have been considered as possible candidates. More recent experiments on ^{176}Os and ^{178}Os [82] indicate that the latter is a good example of X(5). A systematic study [83] of available experimental data on energy levels and B(E2) transition rates suggested ^{126}Ba and ^{130}Ce as possible good candidates, in addition to the N=90 isotones of Nd, Sm, Gd, and Dy. A similar study in lighter nuclei [84] suggested ^{76}Sr , ^{78}Sr and ^{80}Zr as possible candidates. ^{104}Mo has been suggested as a candidate for X(5) based on available spectra [85, 84], but later studies on $B(E2)$ values gave results close to the rigid rotor limit [86]. ^{122}Ba [87] is currently under consideration, since its ground state bands coincides with this of X(5). Recent measurements [88] on ^{128}Ce indicate that this nucleus is a good example of X(5). This is expected, since ^{128}Ce , having 8 valence protons and 12 valence neutron holes, matches ^{152}Sm , possessing 12 valence protons and 8 valence neutrons, which is a good example of X(5).

The assumption of a flat β -potential in the X(5) symmetry has been tested by constructing potential energy surfaces (PESs) for nuclei close to the X(5) symmetry, through the use of relativistic mean field theory [50, 89, 90, 91]. It has been found that the relevant PESs exhibit a bump in the middle, in

accordance to calculations using an effective β deformation, determined by variation after angular momentum projection and two-level mixing [92], as well as in Nilsson-Strutinsky-BCS calculations [93] for ^{152}Sm and ^{154}Gd .

4. Z(5) AND RELATED MODELS

Z(5) [94] is an analogue of X(5) appropriate for triaxial nuclei. In both cases potentials of the form $u(\beta, \gamma) = u(\beta) + u(\gamma)$ are considered. In the X(5) case the Hamiltonian is simplified by focusing attention near $\gamma = 0$, which corresponds to prolate axially symmetric nuclei. In Z(5) attention is focused near $\gamma = \pi/6$, which corresponds to maximally triaxial shapes. It is known [95] that for $\gamma = \pi/6$ the projection of the angular momentum on the body-fixed \hat{x}' -axis, labelled as α , is a good quantum number, while the projection on the body-fixed \hat{z}' -axis, labelled as K , is not a good quantum number. One then seeks solutions of the relevant Schrödinger equation having the form $\Psi(\beta, \gamma, \theta_i) = \phi_\alpha^L(\beta, \gamma) \mathcal{D}_{M, \alpha}^L(\theta_i)$, where θ_i ($i = 1, 2, 3$) are the Euler angles, $\mathcal{D}(\theta_i)$ denote Wigner functions of them, L are the eigenvalues of angular momentum, while M and α are the eigenvalues of the projections of angular momentum on the laboratory-fixed z -axis and the body-fixed x' -axis respectively. α has to be an even integer [95]. Instead of the projection α of the angular momentum on the \hat{x}' -axis, it is customary to introduce the wobbling quantum number [95, 96] $n_w = L - \alpha$.

One is interested in cases near maximal triaxiality, i.e. close to $\gamma = \pi/6$. Thus one uses a harmonic oscillator potential $u(\gamma) = c(\gamma - \frac{\pi}{6})^2/2 = c\tilde{\gamma}^2/2$, with $\tilde{\gamma} = \gamma - \pi/6$.

In the case in which the potential has a minimum around $\gamma = \pi/6$ one can write the last term of Eq. (1) in the form

$$\sum_{k=1,2,3} \frac{Q_k^2}{\sin^2(\gamma - \frac{2\pi}{3}k)} \approx Q_1^2 + 4(Q_2^2 + Q_3^2) = 4(Q_1^2 + Q_2^2 + Q_3^2) - 3Q_1^2. \quad (10)$$

Using this result in the Schrödinger equation corresponding to the Hamiltonian of Eq. (1), introducing [9] reduced energies $\epsilon = 2BE/\hbar^2$ and reduced potentials $u = 2BV/\hbar^2$, and assuming [9] that the reduced potential can be separated into two terms, one depending on β and the other depending on γ , i.e. $u(\beta, \gamma) = u(\beta) + u(\gamma)$, the Schrödinger equation can be separated into two equations [94].

In the equation containing the γ -variable, β^2 denominators remain, which are replaced, in analogy with X(5), by their average values over the β wavefunctions, $\langle \beta^2 \rangle$. Taking into account the simplifications imposed by γ being close to $\pi/6$, the relevant equation takes the form corresponding to a simple

one-dimensional harmonic oscillator in γ , having wavefunctions proportional to Hermite polynomials [94].

The form of the solution of the radial equation depends on the choice made for $u(\beta)$.

4.1. Z(5)

In Z(5) a 5-D infinite well potential is used, as in X(5). The relevant equation is again a Bessel equation, but with order

$$\nu = \frac{\sqrt{4L(L+1) - 3\alpha^2 + 9}}{2} = \frac{\sqrt{L(L+4) + 3n_w(2L - n_w) + 9}}{2}. \quad (11)$$

The solutions still have the form of Eq. (2), with (ξ, τ) replaced by $(s, \nu) = (s, n_w, L)$, where s is the order of the relevant root of the Bessel function $J_\nu(k_{s,\nu}\beta)$. The relevant exactly soluble model is labelled as Z(5) (which is not meant as a group label).

The total energy has the form

$$E(s, n_w, L, n_{\tilde{\gamma}}) = E_0 + A(x_{s,\nu})^2 + Bn_{\tilde{\gamma}}, \quad (12)$$

where $n_{\tilde{\gamma}}$ is the quantum number of the one-dimensional oscillator occurring in the γ -equation, while E_0 , A , B are free parameters.

The wobbling quantum number n_w labels a series of bands with $L = n_w, n_w + 2, n_w + 4, \dots$ (with $n_w > 0$) next to the ground state band (with $n_w = 0$) [95]. The ground state band corresponds to $s = 1$, $n_w = 0$ and has $R_{4/2} = 2.350$. We shall refer to the model corresponding to this solution as Z(5) (which is not meant as a group label), in analogy to the E(5) [8], and X(5) [9] models. Complete level schemes have been given in [94, 97]. A preliminary comparison to experiment has suggested $^{192-196}\text{Pt}$ as possible Z(5) candidates [94, 97].

4.2. OTHER SOLUTIONS

Solutions in the vicinity of $\gamma = \pi/6$ have also been worked out considering potentials of the form $u(\beta, \gamma) = u(\beta) + u(\gamma)/\beta^2$ [98, 99], which allow for an exact separation of variables. Coulomb, Kratzer, harmonic, Davidson, and infinite square well potentials have been considered as $u(\beta)$ in this approach [98, 99], while a displaced harmonic oscillator has been used as $u(\gamma)$. A periodic potential $u(\gamma) = \mu/\sin^2(3\gamma)$ has also been considered [100]. A solution similar to Z(5), but with a $u(\gamma)$ proportional to $\cos^2(3\gamma)$, has also been considered [101].

4.3. Z(4)

The special case in which γ is frozen to $\gamma = \pi/6$, while an infinite square well potential is used in β , leads to an exactly separable four-dimensional model, which has been called Z(4) [102]. This model involves four variables, β and the three Euler angles, since γ in this model is not treated as a variable but as a parameter, as in the model of Davydov and Chaban [103].

Exact separation of variables is possible in this case. The equation involving the Euler angles has been solved by Meyer-ter-Vehn [95], the eigenfunctions being appropriate combinations of Wigner functions. The β -equation, in which an infinite square well potential is used, takes the form of a Bessel equation. The radial solutions have the same form as in Z(5), but with order

$$\nu = \sqrt{L(L+1) - \frac{3}{4}\alpha^2 + 1} = \frac{\sqrt{L(L+4) + 3n_w(2L - n_w) + 4}}{2}, \quad (13)$$

which should be compared to Eq. (11). where the various symbols have the same meaning as in Z(5). The $R_{4/2}$ ratio is 2.226. Complete level schemes have been given in [102, 97]. A preliminary comparison to experiment has suggested $^{128-132}\text{Xe}$ as possible Z(4) candidates [102, 97].

4.4. TRANSITION FROM AXIAL TO TRIAXIAL SHAPES

A special solution of the Bohr Hamiltonian corresponding to a transition from axially deformed to triaxially deformed shapes has been given in Ref. [104], called Y(5). The proton–neutron triaxiality occurring in the $\text{SU}(3)^*$ limit [105] of IBM-2 [3] has triggered the detailed study of the phase structure of IBM-2 [106, 107, 108]. The main features of proton–neutron triaxiality are a low-lying $K = 2$ band and $B(E2)$ s resembling closely these of the Davydov model [109].

5. CONCLUSIONS

A great and still growing interest has been developed in the last five years in special solutions of the Bohr Hamiltonian, in relation to shape phase transitions and critical point symmetries in nuclei. Extensions of these ideas in many directions are ongoing, including the consideration of dipole [47] and octupole [110, 111, 112, 113] degrees of freedom. Many developments relevant to shape phase transitions and critical point symmetries, not mentioned in this work (which is not a review article but rather a biased brief account

of topics related to the authors' work), can be traced from the references in [114, 115, 116].

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