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GAUGE THEORY OF GRAVITATION AND GENERAL RELATIVITY

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Abstract. Some models of the gauge theory for the gravitational interaction are presented and a comparison with the General Relativity is made. The Poincaré and deSitter groups are used as gauge groups and the space-time is chosen to be a Minkowski (flat) one endowed with spherical symmetry. It is shown that these gauge groups can be considered as a "passive" gauge symmetry groups for gravitation. This means that the space-time coordinates are not affected by group transformations. Only the fields change under the action of the symmetry group. It is concluded that the gravitation can be described by gauge potentials defined on a Minkowski space-time and we have not to use Riemann or Riemann-Cartan theories.

Key words: Gauge theory, gravitation, general relativity, Poincaré group, group deSitter.

1. INTRODUCTION

The gauge theories are fundamental in the field theory and, in particular, in the elementary particle physics. The three non-gravitational interactions (electromagnetic, weak, strong) are completely described by means of gauge theories in the framework of the Standard Model (SM).

First of all, the gauge theory of the unitary groups $SU(N)$ is of fundamental importance in elementary particle physics. The SM of strong and electroweak interactions is based on the gauge theory of $SU(3)\times SU(2)\times U(1)$ group. In addition, the "Grand Unification" is described by the gauging of SU(5) group [1].

Secondly, the Poincaré group is also of a fundamental importance in any field theory. After pioneering works of Utiyama [2], Sciama [3, 4], and Kibble [5] it was recognized that gravitation also can be formulated as a gauge theory. It is believed that the formulation of gravity as a gauge theory on a Minkowski space-time could lead to a consistent quantum theory of gravity.

In this Report we present some models of gauge theory for the gravitational interaction and give also a comparison of the corresponding results to those obtained in the framework of the General Relativity (*GR*).

The gauge theory are usually formulated in terms of potentials $A_\mu^a(x)$, where $a = 1, 2, \ldots, m$ (m is the dimension of the gauge group) and $\mu = 0, 1, 2, 3$. In the Lagrangian formalism the equations of the gauge fields $A_\mu^a(x)$ are of second order. In order to simplify the search for solutions of the field equations it is useful to solve equations of first order and this can be done using self-dual (*SD*) models. The self-duality equations are differential equations of the first order and it is easier to investigate the solutions for different particular configurations of the gauge fields and of space-times. We remember that one of the most important property of the self-duality equations is that they imply the Yang-Mills ($Y - M$) field equations [6, 7].

In Section 2 we develope a gauge theory of the Poincaré group P following a model given by MacDowell and Mansouri [8]. This model considers that the gauge group is the de-Sitter group and it enables the construction of ordinary Einstein gravity with or without a cosmological term in four-dimensional space-time. Unlike these authors, we work with the Poincaré group as gauge group on the space-time Minkowski (flat) endowed with spherical symmetry [9, 10]. In this case we have ten potentials, denoted in general form by h_μ^A. We give a gauge theory for gravitation with P as structural group and obtain its structure constants, whose generators are chosen in tensorial notation. Then, we construct a model with spherical symmetry for potentials which has four independent functions, each depending only the $3D$ radius r. For these potentials we obtain the self-duality equations. The *Y–M* equations are written and it is proven that the *SD* equations imply *Y–M* equations. We find also an analytical solution of the model.

Recently, many works have been given with intention to develop a gauge theory of gravitation [11]. Some authors consider the Poincaré group or deSitter group as "active" symmetry groups, *i.e.* acting on the space-time coordinates [12]. Other authors adopt the "passive" point of view when the space-time coordinates are not affected by group transformations [13, 14]. Only the fields change under the action of the symmetry group.

In Section 3 we adopt the second point of view to develop a deSitter (*DS*) gauge theory of gravitation over a spherical symmetric Minkowski space-time. Therefore we restrict ourselves to recast *DS* symmetry and its consequences in the form of an inner symmetry. The coordinate system used to specify the space-time events is not affected anymore by *DS* transformations.

The deSitter gauge model is formulated on a spherical symmetric Minkowski space-time. The general expressions for the components $F_{\mu\nu}^A$ of the strength tensor of the gauge fields are obtained. A particular ansatz for the gauge fields is chosen and the corresponding components $F^{A}_{\mu\nu}$ are calculated. The case of null torsion is also considered and an analytical solution of Scwarzschild-deSitter type is given. The conclusion is that the deSitter group can be considered as a "passive" gauge symmetry group for gravitation. Therefore the gravitation can be described by gauge potentials defined on a Minkowski space-time and we have not to use Riemann or Riemann-Cartan theories.

One fundamental problem in gauge theory of gravitation as well as in *GR* is that of the singularities appearing in different solutions of field equations. In Section 4 we describe a method that enables the obtaining of solutions without singularities for gauge field equations. We use *DS* as gauge group in order to obtain a model with cosmological constant for the gravitational field. The Poincaré gauge theory is obtained as a limit of *DS* model when the cosmological constant vanishes.

First of all, we introduce the gauge fields $e^{a}_{\mu}(x)$ (tetrad fields) and $\omega^{ab}_{\mu}(x)$ (spin connection). They are used to construct the field strengths $F^{a}_{\mu\nu}$ and $F^{ab}_{\mu\nu}$ and the invariants of the theory. Then the integral of the action is written and the constraints for non-singular solutions are introduced in its expression by means of two Lagrange-multiplier fields $\varphi_1(t)$ and $\varphi_1(t)$. The field equations are obtained for a particular form of spherically symmetric gauge fields. They contain the cosmological constant Λ introduced into the model by using the *DS* group as gauge symmetry.

An example of solution without singularities is also presented. This solution is a time-periodic one with frequency of the gravitational field depending on the cosmological constant. It corresponds to a negative value of the cosmological constant ($\Lambda < 0$). The case with positive cosmological constant ($\Lambda > 0$) can be studied choosing the anti-de-Sitter group as gauge group.

In Section 5 we develop a gauge theory for gravitation with *DS* group as structural group and having a spherically symmetric Minkowski space-time as base manifold. We suppose that the gravitational field is created by a point-like source of mass *m* which has also an electrical charge *Q*. The gravitational field is described by the gauge potentials associated to the *DS* group and the effect of the electrical charge *Q* is included in model by energy-momentum tensor of the corresponding electromagnetic field [30, 32]. We construct the integrals of action S_g and S_{em} of the gravitational and respectively electromagnetic field. The field equations are obtained by imposing the variational principle $\delta S = 0$ on the total integral of action $S = S_g + S_{em}$ associated to the system composed of the two fields. As an example, a model with spherical symmetry is constructed. It is shown that one of the solutions of field equations includes simultaneously the Schwarzschild, Reissner-Nordström and de-Sitter ones.

2. SELF-DUAL POINCARÉ GAUGE THEORY

2.1. THE GAUGE THEORY OF GRAVITATION

We will present a model for a self-dual gauge theory of the Poincaré group P in the 4-dimensional space-time Minkowski, endowed with spherical symmetry:

$$\mathrm{d}s^2 = \mathrm{d}t^2 - \mathrm{d}r^2 - r^2\mathrm{d}\theta^2 - r^2 \sin^2\theta\, \mathrm{d}\varphi^2 \tag{2.1}$$

The group P is 10-dimensional. For the infinitesimal generators of the group P we use the tensorial notation P_a, $M_{ab} = -M_{ba}$, $a = 0, 1, 2, 3$, where P_a generate the 4-dimensional group of the space-time translations and M_{ab} generate the Lorentz algebra. Then, the structure equations of Poincaré group P are [8]:

$$\left[P_a, P_b\right] = 0, \tag{2.2a}$$

$$\left[P_a, M_{bc}\right] = i\left(\eta_{ab}P_c - \eta_{ac}P_b\right), \tag{2.2b}$$

$$\left[M_{ab}, M_{cd}\right] = i\left(\eta_{ad}M_{bc} - \eta_{ac}M_{bd} + \eta_{bc}M_{ad} - \eta_{bd}M_{ac}\right). \tag{2.2c}$$

In order to give a general formulation of the gauge theory for the Poincaré group, we will denote the ten generators P_a and M_{ab} by X_A, $A = 1, 2, \ldots 10$. Then, the equations of structure (2.2) can be written in the general form:

$$\left[X_A, X_B\right] = i f_{AB}^{C} X_C, \tag{2.3}$$

where f_{AB}^{C} are the constants of structure whose concrete expressions will be given below (see eqs. (2.6)).

We suppose now that the Poincaré group P is a gauge group for gravitation and we introduce 10 gauge fields $h_\mu^A(x)$, $A = 1, 2, \ldots, 10$, $\mu = 0, 1, 2, 3$. Then, we construct the tensor of the gauge fields $F_{\mu\nu} = F_{\mu\nu}^A X_A$, which have the values in the Lie algebra of Poincaré group. The components of this tensor are given by [9, 10]:

$$F_{\mu\nu}^A = \partial_\mu h_\nu^A - \partial_\nu h_\mu^A + f_{BC}^A h_\mu^B h_\nu^C . \tag{2.4}$$

In order to write the structure constants f_{AB}^{C}, we use the following notation for the index A:

$$A = \begin{cases} a = 0, 1, 2, 3; \\ [bc] = [01],[02],[03],[12],[13],[23] \end{cases} \tag{2.5}$$

This means that we have $X_{[bc]} = M_{bc}$ and $X_a = P_a$ in the relations (2.2). From the relations (2.2), (2.3) and (2.5) we find the following expressions of the structure constants:

$$f_{bc}^{a} = f_{c[de]}^{[ab]} = f_{[bc][de]}^{a} = f_{cd}^{[ab]} = 0, \tag{2.6a}$$

$$f_{b[cd]}^{a} = -f_{[cd]b}^{a} = \frac{1}{2}\left(\eta_{bc}\delta_{d}^{a} - \eta_{bd}\delta_{c}^{a}\right), \tag{2.6b}$$

$$f_{[ab][cd]}^{[ef]} = \frac{1}{4}\left(\eta_{bc}\delta_{a}^{e}\delta_{d}^{f} - \eta_{ac}\delta_{b}^{e}\delta_{d}^{f} + \eta_{ad}\delta_{b}^{e}\delta_{c}^{f} - \eta_{bd}\delta_{a}^{e}\delta_{c}^{f}\right) - e \leftrightarrow f. \tag{2.6c}$$

Here $\eta_{ab} = \mathrm{diag}(1, -1, -1, -1)$ is the Minkowski metric used on the Poincaré group manifold and δ_{b}^{a} is the usual Kronecker symbol.

Now, in order to give a geometrical significance of the gauge field tensor, we make the following notational changes for the gauge fields:

$$h_{\mu}^{a} \equiv e_{\mu}^{a} \qquad \text{and} \qquad h_{\mu}^{[ab]} \equiv \omega_{\mu}^{ab}, \quad \omega_{\mu}^{ab} = -\omega_{\mu}^{ba}. \tag{2.7}$$

The e_{μ}^{a} gauge fields will be interpreted as usual vierbein (or tetrads) and ω_{μ}^{ab} as the spin connection (Ricci rotation coefficients) [8]. Inserting the relation (2.6) and (2.7) into the definition (2.4), we find the expressions of the stress-tensor components:

$$F_{\mu\nu}^{a} = \partial_{\mu}e_{\nu}^{a} - \partial_{\nu}e_{\mu}^{a} + \left(\omega_{\mu}^{ab}e_{\nu}^{c} - \omega_{\nu}^{ab}e_{\mu}^{c}\right)\eta_{bc}, \tag{2.8}$$

and

$$F_{\mu\nu}^{ab} = \partial_{\mu}\omega_{\nu}^{ab} - \partial_{\nu}\omega_{\mu}^{ab} + \left(\omega_{\mu}^{ac}\omega_{\nu}^{db} - \omega_{\nu}^{ac}\omega_{\mu}^{db}\right)\eta_{cd} \equiv R_{\mu\nu}^{ab}. \tag{2.9}$$

For the gauge-covariant derivative we have used the form:

$$D_{\mu} = \partial_{\mu} - i\left[h_{\mu}, \cdot\right]. \tag{2.10}$$

The quantity $F_{\mu\nu}^{a}$ is the torsion tensor and can written as:

$$F_{\mu\nu}^{a} \equiv T_{\mu\nu}^{a} = D_{\mu}e_{\nu}^{a} - D_{\nu}e_{\mu}^{a} \tag{2.11}$$

with

$$D_{\mu}e_{\nu}^{a} = \partial_{\mu}e_{\nu}^{a} + \omega_{\mu}^{ab}e_{\nu}^{c}\eta_{bc}, \tag{2.12}$$

and the quantity $F_{\mu\nu}^{ab}$ is the curvature tensor.

2.2. GAUGE POTENTIALS AND SELF-DUALITY EQUATIONS

We consider a particular form of spherically gauge fields of the Poincaré group given by the following ansatz:

$$e_\mu^0 = (A, 0, 0, 0), \quad e_\mu^1 = \left(0, \frac{1}{r^2 A}, 0, 0\right), \tag{2.13a}$$

$$e_\mu^2 = (0, 0, rC, 0), \quad e_\mu^3 = (0, 0, 0, rC\sin\theta), \tag{2.13b}$$

and

$$\omega_\mu^{01} = (U, 0, 0, 0), \quad \omega_\mu^{02} = \omega_\mu^{03} = \omega_\mu^{12} = \omega_\mu^{13} = (0, 0, 0, 0), \tag{2.14a}$$

$$\omega_\mu^{23} = (iV, 0, 0, \cos\theta), \tag{2.14b}$$

where A, C, U and V are functions only of the $3D$ radius r. We use the above expressions to compute the components of the tensors $F_{\mu\nu}^a$ and $F_{\mu\nu}^{ab}$. The non-null components $F_{\mu\nu}^a$ and, respectively, $F_{\mu\nu}^{ab}$ are:

$$F_{01}^0 = -\frac{r^2 AA' + U}{r^2 A}, \quad F_{03}^2 = -irCV\sin\theta, \quad F_{12}^2 = C + rC', \tag{2.15a}$$

$$F_{02}^3 = irCV, \quad F_{13}^3 = (C + rC')\sin\theta, \tag{2.15b}$$

$$F_{01}^{01} = -U', \quad F_{01}^{23} = -iV', \quad F_{23}^{23} = -\sin\theta, \tag{2.15c}$$

where A', C', U', V' denote the derivatives with respect to the variable r.

In order to obtain a self-dual model, first of all, we define the dual tensor ${}^*F_{\mu\nu}$ by [6, 11]:

$${}^*F_{\mu\nu} = \frac{1}{2}\sqrt{-g}\,\varepsilon_{\mu\nu\rho\sigma}F^{\rho\sigma}, \tag{2.16}$$

where "*" is the Hodge dual map and $\varepsilon_{\mu\nu\rho\sigma}$ is the Levi-Civita symbol of rank four, with $\varepsilon_{0123} = 1$. In our case, the components of the dual tensor ${}^*F_{\mu\nu}$ are:

$${}^*F_{\mu\nu}^a = \frac{1}{2}\sqrt{-g}\,\varepsilon_{\mu\nu\rho\sigma}F^{a\rho\sigma} \qquad \text{and} \qquad {}^*F_{\mu\nu}^{ab} = \tfrac{1}{2}\sqrt{-g}\,\varepsilon_{\mu\nu\rho\sigma}F^{ab\rho\sigma}. \tag{2.17}$$

We obtain the following non-null components of the dual tensor ${}^*F_{\mu\nu}^a$, respectively ${}^*F_{\mu\nu}^{ab}$:

$${}^*F_{\mu\nu}^{ab}:\ {}^*F_{23}^0 = \frac{r^2 AA' + U}{A}\sin\theta, \quad {}^*F_{03}^2 = (C + rC')\sin\theta, \quad {}^*F_{12}^2 = irCV, \tag{2.18a}$$

$${}^*F_{02}^3 = -C - rC', \qquad {}^*F_{13}^3 = irCV\sin\theta, \tag{2.18b}$$

$${}^*F_{23}^{01} = r^2 U'\sin\theta, \qquad {}^*F_{01}^{23} = -\frac{1}{r^2}, \qquad {}^*F_{23}^{23} = ir^2 V'\sin\theta. \tag{2.18c}$$

The *Y-M* field equations are solved by any gauge fields, which satisfy the self-duality condition [6, 11]:

$$ {}^{*}F_{\mu\nu} = iF_{\mu\nu}. \tag{2.19}$$

From the relation (2.19) it follows:

$$ {}^{*}F^{a}_{\mu\nu} = iF^{a}_{\mu\nu}, \tag{2.20}$$

$$ {}^{*}F^{ab}_{\mu\nu} = iF^{ab}_{\mu\nu}. \tag{2.21}$$

Now, we introduce the expressions (2.15) and (2.18) in the expressions (2.20) and (2.21) and then we write the self-duality equations. For the first set of equations (2.20), we obtain only two independent equations:

$$A' + \frac{U}{r^2 A} = 0, \tag{2.22a}$$

$$rC' + \left(1 - rV\right)C = 0. \tag{2.22b}$$

The second set of equations (2.21) reduces too only to the following two independent equations:

$$U' = 0, \tag{2.23a}$$

$$V' = -\frac{1}{r^2}. \tag{2.23b}$$

The equations (2.22) and (2.23) are self-duality equations on the Minkowski space-time endowed with spherical symmetry and with the Poincaré group as gauge group. We remark that these equations are of the first order unlike the *Y-M* equations which are of the second order. From this reason, the search of solutions is easier. We remember that, for the Minkowski space-time, the solutions of self-duality equations are automatically solutions for the *Y-M* equations [9].

2.3. THE YANG-MILLS EQUATIONS

In this subsection we obtain the *Y-M* equations for the gauge fields h^A_μ and we prove that these can be obtained starting from the self-duality equations. The field equations for the gauge fields h^A_μ can be written in the form [7]:

$$\frac{1}{\sqrt{-g}}\partial_\mu\left(\sqrt{-g}F^{A\mu\nu}\right) + f^A_{BC}h^B_\mu F^{C\mu\nu} = 0. \tag{2.24}$$

In our case, using the notation (2.5) for index A and the structure constants (2.6) we obtain two sets, denoted by $E^{a\nu}$ and $E^{ab\nu}$, for the *Y-M* equations:

$$E^{a\nu} \equiv \frac{1}{\sqrt{-g}}\partial_\mu\left(\sqrt{-g}F^{a\mu\nu}\right) + f^a_{b[cd]}e^b_\mu F^{cd\mu\nu} + f^a_{[bc]d}\omega^{bc}_\mu F^{d\mu\nu} = 0 \tag{2.25}$$

$$E^{ab\nu} \equiv \frac{1}{\sqrt{-g}}\partial_\mu\left(\sqrt{-g}F^{ab\mu\nu}\right) + f^{[ab]}_{[cd][ef]}\omega^{cd}_\mu F^{ef\mu\nu} = 0. \tag{2.26}$$

From here, we find only four independent equations, which are:

$$A'' + \frac{2A'}{r} + \frac{2U'}{r^2 A} - \frac{UA'}{r^2 A^2} = 0, \tag{2.27a}$$

$$r^2 C'' + 2rC' + \left(1 - r^2 V^2\right)C = 0, \tag{2.27b}$$

$$AU' - UA' - \frac{U^2}{r^2 A} = 0, \tag{2.27c}$$

$$rV'' + 2V' = 0. \tag{2.27d}$$

It is easy to verify that from the equations (2.27) it results the following equivalent field equation for the gauge potential U:

$$rU'' + 2U' = 0. \tag{2.28}$$

The equations (2.27) are the *Y-M* equations for the our ansatz. These equations can be obtained from the self-duality equations if we derive the first equation from (22) after r, respectively the second equation, and using the equations (23). By consequence, we proved that the solutions of the self-duality equations are and solutions for the *Y-M* equations.

2.4. EXACT SOLUTIONS

We are going now to find the exact solutions with spherical symmetry for the field equations (2.22) and (2.23). From the equations (2.23) we obtain, by integration, the following solutions for U and V:

$$U = \alpha, \qquad V(r) = \frac{1}{r} + \beta, \tag{2.29}$$

where α and β are two arbitrary constants of integration. Inserting the expressions of functions U and V in the equations (2.21) we obtain:

$$A(r) = \sqrt{a + \frac{2\alpha}{r}}, \qquad C(r) = be^{\beta r}, \tag{2.30}$$

a and b being other two integration constants.

The gauge potentials e_{μ}^{a} and $\omega_{\mu\nu}^{ab}$, which describe the gravitational field, become:

$$e_{\mu}^{0}=\left(\sqrt{a+\frac{2\alpha}{r}},0,0,0\right),\quad e_{\mu}^{1}=\left(0,\frac{1}{r^{2}\sqrt{a+\frac{2\alpha}{r}}},0,0\right), \tag{2.31a}$$

$$e_{\mu}^{2}=\left(0,0,bre^{\beta r},0\right),\quad e_{\mu}^{3}=\left(0,0,0,bre^{\beta r}\sin\theta\right), \tag{2.31b}$$

and, respectively,

$$\omega_{\mu}^{01}=\left(\alpha,0,0,0\right),\quad \omega_{\mu}^{02}=\omega_{\mu}^{03}=\omega_{\mu}^{12}=\omega_{\mu}^{13}=\left(0,0,0,0\right), \tag{2.32a}$$

$$\omega_{\mu}^{23}=\left(i\left(\frac{1}{r}+\beta\right),0,0,\cos\theta\right). \tag{2.32b}$$

If we define, as usually, a new metric $\overline{g}$ by the formula:

$$\overline{g}_{\mu\nu}=e_{\mu}^{a}e_{\nu}^{b}\eta_{ab}, \tag{2.33}$$

then we obtain, in our case, the following non-null metric coefficients:

$$\overline{g}_{00}=a+\frac{2\alpha}{r},\quad \overline{g}_{11}=\frac{1}{r^{4}\left(a+\frac{2\alpha}{r}\right)},\quad \overline{g}_{22}=b^{2}r^{2}e^{2\beta r},\quad \overline{g}_{33}=b^{2}r^{2}e^{2\beta r}\sin^{2}\theta. \tag{2.34}$$

The corresponding expression for the square of the line element is:

$$d\sigma^{2}=\left(a+\frac{2\alpha}{r}\right)dt^{2}-\frac{1}{r^{4}\left(a+\frac{2\alpha}{r}\right)}dr^{2}-b^{2}r^{2}e^{2\beta r}\left(d\theta^{2}+\sin^{2}\theta\, d\varphi^{2}\right). \tag{2.35}$$

In particular, if we choose the constants of integration α, β, a and b equal to:

$$\alpha=-m,\quad \beta=0,\quad a=1,\quad b=1,$$

then the line element (2.35) becomes:

$$d\sigma^{2}=\left(1-\frac{2m}{r}\right)dt^{2}-\frac{1}{r^{4}\left(1-\frac{2m}{r}\right)}dr^{2}-r^{2}\left(d\theta^{2}+\sin^{2}\theta\, d\varphi^{2}\right). \tag{2.36}$$

This expression can be considered as an interior solution for the gravitational field, that means it is valid only in the region $0<r<2m$.

Considering the metric $\overline{g}$ and the torsion $F_{\mu\nu}^{a}$ given by the formula (2.11), we can develop a Riemann-Cartan theory of the gravitational field.

3. DE-SITTER GAUGE THEORY

We will generalize now the Poincaré gauge theory to the case when the structural group is *DS* group. In this case we obtain a gauge theory of gravitation with the cosmological constant Λ appearing in the field equations and their solutions. Therefore, the corresponding theory is adequate to describe cosmological models.

3.1. MODEL WITH DESITTER GAUGE GROUP

We develop a gauge theory of the deSitter group *DS* in a 4-dimensional Minkowski space-time, endowed with the spherical symmetry (2.1) The *DS* group is also 10-dimensional and its infinitesimal generators are denoted by P_a and $M_{ab} = -M_{ba}$, $a, b = 0, 1, 2, 3$ [7, 15, 16]. We denote these generators by X_A, $A = 1, 2, \ldots 10$. Then, the equations of structure for the *DS* group can be written under the general form (2.3), where $f_{AB}^{C} = -f_{AB}^{C}$ are the constants of structure of *DS* group and they have expressions that differ from those of the Poincaré group [see eq. (3.1)].

Let us suppose now that the deSitter group *DS* is a gauge group for gravitation; corresponding, we introduce 10 gauge fields $h_\mu^A(x)$, $A = 1, 2, \ldots, 10$, $\mu = 0, 1, 2, 3$. Then, we construct the tensor of the gauge fields (strength tensor) $F_{\mu\nu} = F_{\mu\nu}^A X_A$ which takes its values in the Lie algebra of the deSitter group *DS* (Lie algebra-valued tensor). The components of this tensor are given by the same general formula (2.4) as in the case of Poincaré group. In order to write the constants of structure f_{AB}^C for *DS* group, we also use the notation (2.5) for the index *A*. This means that we have $X_a = P_a$, $X_{[bc]} = M_{ab}$. For the constant of structure we find now the following expressions:

$$\begin{aligned}
f_{bc}^{a} &= f_{c[de]}^{[ab]} = f_{[bc][de]}^{a} = 0, \\
f_{cd}^{[ab]} &= 4\lambda^2\left(\delta_c^b\delta_d^a - \delta_c^a\delta_d^b\right) = -f_{dc}^{[ab]}, \\
f_{b[cd]}^{a} &= -f_{[cd]b}^{a} = \frac{1}{2}\left(\eta_{bc}\delta_d^a - \eta_{bd}\delta_c^a\right), \\
f_{[ab][cd]}^{[ef]} &= \frac{1}{4}\left(\eta_{bc}\delta_a^e\delta_d^f - \eta_{ac}\delta_b^e\delta_d^f + \eta_{ad}\delta_b^e\delta_c^f - \eta_{bd}\delta_a^e\delta_c^f\right) - e \longleftrightarrow f,
\end{aligned} \tag{3.1}$$

where λ is a deformation parameter. When $\lambda \to 0$, we obtain the Poincaré Lie algebra.

We still denote the gauge fields $h_\mu^A(x)$ by $e_\mu^a(x)$ (tetrad fields) if $A = a$ and by $\omega_\mu^{ab}(x) = -\omega_\mu^{ba}(x)$ (spin connection) if $A = [ab]$. Then, introducing the relations (3.1) into the definition (2.4), we find the following expressions of the strength tensor components:

$$F^a_{\mu\nu} = \partial_\mu e^a_\nu - \partial_\nu e^a_\mu + \left(\omega^{ab}_\mu e^c_\nu - \omega^{ab}_\nu e^c_\mu\right)\eta_{bc} = T^a_{\mu\nu}, \tag{3.2}$$

$$F^{ab}_{\mu\nu} = \partial_\mu \omega^{ab}_\nu - \partial_\nu \omega^{ab}_\mu + \left(\omega^{ac}_\mu \omega^{db}_\nu - \omega^{ac}_\nu \omega^{db}_\mu\right)\eta_{cd} - 4\lambda^2\left(e^a_\mu e^b_\nu - e^a_\nu e^b_\mu\right) = R^{ab}_{\mu\nu} \tag{3.3}$$

The quantity $F^a_{\mu\nu} = T^a_{\mu\nu}$ is interpreted as the torsion tensor and $F^{ab}_{\mu\nu} = R^{ab}_{\mu\nu}$ as the curvature tensor of a Riemann-Cartan space-time defined by the gravitational gauge fields e^a_μ and ω^{ab}_μ.

3.2. THE STRENGTH TENSOR OF THE GAUGE FIELDS

We consider a particular form of spherically gauge fields of the deSitter group DS given by the following ansatz:

$$\begin{aligned} &e^0_\mu = (A, 0, 0, 0), \quad e^1_\mu = \left(0, \frac{1}{A}, 0, 0\right), \\ &e^2_\mu = (0, 0, rC, 0), \quad e^3_\mu = (0, 0, 0, rC\sin\theta) \end{aligned} \tag{3.4}$$

$$\begin{aligned} &\omega^{01}_\mu = (U, 0, 0, 0), \quad \omega^{12}_\mu = (0, 0, W, 0), \quad \omega^{13}_\mu = (0, 0, 0, Z\sin\theta), \\ &\omega^{23}_\mu = (V, 0, 0, \cos\theta), \quad \omega^{02}_\mu = \omega^{03}_\mu = (0, 0, 0, 0), \end{aligned} \tag{3.5}$$

where A, C, U, V, W, and Z are functions only of the $3D$ radius r. We use the above expressions to compute the components of the tensors $F^a_{\mu\nu}$ and $F^{ab}_{\mu\nu}$. The non-null components of $F^a_{\mu\nu}$ and $F^{ab}_{\mu\nu}$ are:

$$\begin{aligned} &F^0_{01} = -\frac{AA' + U}{A}, \quad F^2_{03} = -rCV\sin\theta, \quad F^2_{12} = C + rC' - \frac{W}{A}, \\ &F^3_{02} = rCV, \quad F^3_{13} = \left(C + rC' - \frac{Z}{A}\right)\sin\theta, \end{aligned} \tag{3.6}$$

respectively

$$\begin{aligned} &F^{01}_{01} = -U' - 4\lambda^2, \quad F^{23}_{01} = -V', \quad F^{13}_{23} = (Z - W)\cos\theta, \\ &F^{02}_{02} = -WU - 4\lambda^2 rCA, \quad F^{13}_{02} = VW, \\ &F^{03}_{03} = -(ZU + 4\lambda^2 rCA)\sin\theta, \quad F^{12}_{12} = W' - 4\lambda^2 r\frac{C}{A}, \\ &F^{12}_{03} = -VZ\sin\theta, \quad F^{13}_{13} = \left(Z' - 4\lambda^2 r\frac{C}{A}\right)\sin\theta, \\ &F^{23}_{23} = -(1 - ZW + 4\lambda^2 r^2 C^2)\sin\theta. \end{aligned} \tag{3.7}$$

3.3. THE CASE OF NULL TORSION

We will develop an Einstein model for gravitation, *i.e.*, we will suppose that the torsion vanishes. First of all, we calculate the Riemann tensor of the model, defined by the formula [18]:

$$\tilde{R}^{\rho\sigma}_{\mu\nu} = F^{ab}_{\mu\nu} e^{\rho}_{a} e^{\sigma}_{b}; \tag{3.8}$$

the corresponding non-null components are:

$$\begin{aligned}
&\tilde{R}^{01}_{01} = -U' - 4\lambda^2, && \tilde{R}^{23}_{01} = -\frac{V'}{r^2C^2\sin\theta},\\
&\tilde{R}^{02}_{02} = -\frac{WU}{rCA} - 4\lambda^2, && \tilde{R}^{13}_{02} = \frac{AVW}{rC\sin\theta},\\
&\tilde{R}^{03}_{03} = -\frac{ZU}{rCA} - 4\lambda^2, && \tilde{R}^{12}_{03} - \frac{AVZ\sin\theta}{rC},\\
&\tilde{R}^{12}_{12} = \frac{AW'}{rC} - 4\lambda^2, && \tilde{R}^{13}_{13} = \frac{AZ'}{rC} - 4\lambda^2,\\
&\tilde{R}^{13}_{23} = \frac{(Z-W)A\cos\theta}{rC\sin\theta}, && \tilde{R}^{23}_{23} = \frac{ZW-1}{r^2C^2} - 4\lambda^2.
\end{aligned} \tag{3.9}$$

Then, we calculate the components of the Ricci tensor, defined as:

$$\tilde{R}^{\nu}_{\mu} = R^{ab}_{\mu\rho} e^{\nu}_{a} e^{\rho}_{b} = \tilde{R}^{\nu\rho}_{\mu\rho}, \tag{3.10}$$

where the sum over the index $\rho = 0, 1, 2, 3$ is understand in the last equality. Using (3.10) we obtain the following non-null components:

$$\begin{aligned}
\tilde{R}^{\nu}_{\mu} &= -U' - \frac{WU}{rCA} - \frac{ZU}{rCA} - 12\lambda^2,\\
\tilde{R}^{1}_{1} &= -U' + \frac{AW'}{rC} + \frac{AZ'}{rC} - 12\lambda^2,\\
\tilde{R}^{1}_{2} &= \frac{(Z-W)A\cos\theta}{rC\sin\theta},\\
\tilde{R}^{2}_{2} &= -\frac{WU}{rCA} + \frac{W'A}{rC} - \frac{1-ZW}{r^2C^2} - 12\lambda^2,\\
\tilde{R}^{3}_{3} &= -\frac{ZU}{rCA} + \frac{Z'A}{rC} - \frac{1-ZW}{r^2C^2} - 12\lambda^2.
\end{aligned} \tag{3.11}$$

In order to write the equations of Einstein, we calculate the curvature scalar $\tilde{R} = \tilde{R}^{\mu}_{\mu}$ (sum over $\mu = 0, 1, 2, 3$). The expression of $\tilde{R}$ in our model is:

$$\tilde{R} = -2\left(U' + \frac{WU}{rCA} - \frac{AW'}{rC} + \frac{1-ZW}{r^2C^2} + \frac{ZU}{rCA} - \frac{Z'A}{rC} + 24\lambda^2\right). \tag{3.12}$$

The equations of Einstein for the vacuum can be written in the form:

$$\tilde{R}_{\mu}^{\nu} - \frac{1}{2}\delta_{\mu}^{\nu}\tilde{R} = 0. \tag{3.13}$$

For the above considered model, these equations become:

$$\begin{gathered}
-\frac{AW'}{rC} + \frac{1 - ZW}{r^2C^2} - \frac{AZ'}{rC} + 12\lambda^2 = 0, \\
\frac{WU}{rC} + \frac{1 - ZW}{r^2C^2} + \frac{ZU}{rC} + 12\lambda^2 = 0, \\
U' + \frac{ZU}{rCA} - \frac{AZ'}{rC} + 12\lambda^2 = 0, \\
U' + \frac{WU}{rCA} - \frac{AW'}{rC} + 12\lambda^2 = 0, \\
(W - Z)A = 0.
\end{gathered} \tag{3.13}$$

If we consider the case of the model with null torsion, then, from (3.6) we obtain the following constraints:

$$U = -AA', \quad V = 0, \quad W = Z = A(C + rC'). \tag{3.14}$$

For these constraints, and imposing the supplementary condition $C = 1$, we obtain only two independent equations:

$$\begin{gathered}
-\frac{2AA'}{r} + \frac{1 - A^2}{r^2} + 12\lambda^2 = 0, \\
-\frac{2AA'}{r} + U' + 12\lambda^2 = 0.
\end{gathered} \tag{3.15}$$

3.4. SOLUTION OF SCWARZSCHILD-DESITTER TYPE

The equations (3.15) are compatible if we chose the function $A(r)$ so that:

$$U' = \frac{1 - A^2}{r^2}. \tag{3.16}$$

Tacking into account first condition (3.14), the equation (3.16) becomes:

$$-(AA') = \frac{1 - A^2}{r^2}, \tag{3.17}$$

or, equivalently

$$r^2\left(A^2\right)'' - 2A^2 + 2 = 0. \tag{3.18}$$

The solution of the equation (3.18) is the following:

$$A^2 = 1 + \frac{\alpha}{r} + \beta r^2, \tag{3.19}$$

where α and β are two arbitrary constants. This solution verifies also the equations of Einstein (3.15) if and only if $\beta = 4\lambda^2$. But, according to the result of MacDowell-Mansouri [7], the cosmological constant of the model is identified as $\Lambda = -12\lambda^2$. Then, the solution (24) have the form:

$$A^2 = 1 + \frac{\alpha}{r} - \frac{\Lambda}{3} r^2. \tag{3.20}$$

In particular, if we chose $\alpha = -2M$, then we obtain the Scwarzschild-deSitter solution:

$$A^2 = 1 - \frac{2M}{r} - \frac{\Lambda}{3} r^2. \tag{3.21}$$

In the limit $\lambda \to 0$, we obtain the Schwarzschild solution:

$$A^2 = 1 - \frac{2M}{r}, \tag{3.22}$$

and for $\alpha = 0$ the solution (3.21) is that of deSitter.

The solution (3.20) corresponds to a metric

$$g_{\mu\nu} = e_\mu^a e_\nu^b \eta_{ab} \tag{3.23}$$

of the form

$$\mathrm{d}s^2 = \left(1 - \frac{2M}{r} - \frac{\Lambda}{3} r^2\right)\mathrm{d}t^2 - \frac{\mathrm{d}r^2}{1 - \frac{2M}{r} - \frac{\Lambda}{3} r^2} - r^2\left(\mathrm{d}\theta^2 + \sin^2\theta\mathrm{d}\varphi^2\right). \tag{3.24}$$

Therefore, the gauge fields e_μ^a can be interpreted as the tetrad fields of the gravitational field. The spin connection components are determined by tetrads e_μ^a (*i.e.* they are not independent fields):

$$U = -\frac{M}{r^2} + \frac{\Lambda}{3} r, \quad W = Z = \left(1 - \frac{2M}{r} - \frac{\Lambda}{3} r^2\right)^{1/2}. \tag{3.25}$$

The previous results show that the model presented in this paper can be considered as a gauge theory for the “active” deSitter symmetry group.

4. SOLUTIONS WITHOUT SINGULARITIES

4.1. GAUGE MODEL OF GRAVITATION

We consider a gauge theory of gravitation having *DS* group as local symmetry. Let X_A, $A = 1, 2, \ldots, 10$, be a basis of *DS* Lie algebra with the corresponding equations of structure given by [17, 18]. We envision the space-time as a four-dimensional manifold M_4; at each point of M_4 we have a copy of *DS* group (*i.e.*, a fibre, in fibre-bundle terminology). Introduce, as usually, the gauge potentials $h_\mu^A(x)$, $A = 1, 2, \ldots, 10$, $\mu = 0, 1, 2, 3$, where (x) denotes the local coordinates on M_4. Then, we calculate the field-strengths $F_{\mu\nu} = F_{\mu\nu}^A X_A$, which take values in Lie algebra of *DS* group (Lie-algebra valued).

The integral of action associated to the gravitational gauge fields, quadratic in the components $F_{\mu\nu}^A$, is written in the form [7, 24]:

$$S_g = \int \mathrm{d}^4 x \varepsilon^{\mu\nu\rho\sigma} F_{\mu\nu}^A F_{\rho\sigma}^B Q_{AB}, \tag{4.1}$$

where $\varepsilon^{\mu\nu\rho\sigma}$ is the Levi-Civita symbol of rank four, with $\varepsilon^{0123} = 1.$ This action is independent of any specific metric on M_4; indeed, the property of general covariance for action imposes the volume element $\sqrt{-g}d^4x$ [with $g = \det(g_{\mu\nu})$] and the tensor Levi-Civita has the form $\frac{1}{\sqrt{-g}}\varepsilon^{\mu\nu\rho\sigma}$, so that the $g_{\mu\nu}$-dependence of S_g cancels.

The quantities Q_{AB} are constants symmetric with respect to the indices A, B: $Q_{AB} = Q_{BA}$. If we chose [7]

$$Q_{AB} = \begin{cases} \varepsilon_{abcd}, & \text{for} \quad A = [ab],\ B = [cd], \\ 0 & \text{otherwise}, \end{cases} \tag{4.2}$$

then we obtain the action integral of *GR*. It is possible also to obtain the integral action of Teleparallel Gravity (*TG*) by an appropriate choice [20, 21] of Q_{AB}.

Now, we use the form given in Eq. (4.2) in order to obtain solutions without singularities of *DS*-gauge theory of gravitation. Namely, we impose some restrictions [22] on two invariants I_1 and I_2 of the theory. Introducing the Lagrange-multiplier $\varphi_1(t)$ and $\varphi_2(t)$, and using the choice (4.2), the integral of action (4.1) can be rewritten as:

$$S_g = -\frac{1}{16\pi G}\int \mathrm{d}^4 x e\left[F + \varphi_1(t) f_1(I_1) + \varphi_2(t) f_2(I_2) + V(\varphi_1, \varphi_2)\right], \tag{4.3}$$

where

$$F = F^{ab}_{\mu\nu}\bar{e}^{\mu}_{a}\bar{e}^{\nu}_{b}, \quad e = \det\left(e^{a}_{\mu}\right). \tag{4.4}$$

and $\bar{e}^{\nu}_{b}$ – the inverse of e^{a}_{μ} defined by Eq. (4.14) below. The quantities $f_i\left(I_i\right)$, $i = 1,2$ are functions which must be chosen in an appropriate form in order to obtain solutions without singularities of the corresponding field equations. Thus, the potential $V\left(\varphi_1, \varphi_2\right)$ have to satisfy the constraint equations [22, 24]:

$$f_1\left(I_1\right) = -\frac{\partial V}{\partial \varphi_1}, \quad f_2\left(I_2\right) = -\frac{\partial V}{\partial \varphi_2}, \tag{4.5}$$

The model can be simplified further if we assume:

$$V\left(\varphi_1, \varphi_2\right) = V_1\left(\varphi_1\right) + V_2\left(\varphi_2\right), \tag{4.6}$$

and chose the invariants I_1, I_2 in the form

$$I_1 = F - \sqrt{3}\left(4F^{a}_{\mu}F^{\mu}_{a} - F^2\right)^{1/2}, \tag{4.7}$$

respectively

$$I_2 = 4F^{a}_{\mu}F^{\mu}_{a} - F^2. \tag{4.8}$$

In these expressions, the quantities F^{a}_{μ} are defined by

$$F^{a}_{\mu} = F^{ab}_{\mu\nu}\bar{e}^{\nu}_{b}. \tag{4.9}$$

As an example, we chose the functions f_1 and f_2 in the simple form [22, 24]:

$$f_1\left(I_1\right) = I_1, \quad f_2\left(I_2\right) = -\sqrt{I_2}. \tag{4.10}$$

Then, the action S_g in Eq. (4.3) becomes:

$$S_g = -\frac{1}{16\pi G}\int \mathrm{d}^4 x e\left[F + \varphi_1 I_1 - \varphi_2\sqrt{I_2} + V_1\left(\varphi_1\right) + V_2(\varphi_2)\right]. \tag{4.11}$$

Now, all we have to do is to write the variational field equations which follow from (4.11) and search their solutions without singularities.

4.2. FIELD EQUATIONS

We develop the *DS* gauge theory in a space-time Minkowski M_4 endowed with spherical symmetry, having the metric given in (2.1). In addition, we choose a particular form of spherically gauge fields $e^{a}_{\mu}\left(x\right)$ and $\omega^{ab}_{\mu}\left(x\right)$ given by the following ansatz:

$$e_\mu^0 = \left(N(t), 0, 0, 0\right), \quad e_\mu^1 = \left(0, \frac{a(t)}{\sqrt{1-kr^2}}, 0, 0\right), \tag{4.12a}$$

$$e_\mu^2 = \left(0, 0, ra(t), 0\right), \quad e_\mu^2 = \left(0, 0, 0, ra(t)\sin\theta\right), \tag{4.12b}$$

respectively

$$\omega_\mu^{01} = \left(0, -\frac{\dot{a}(t)}{N(t)\sqrt{1-kr^2}}, 0, 0\right), \quad \omega_\mu^{02} = \left(0, 0, -\frac{r\dot{a}(t)}{N(t)}, 0\right), \tag{4.13a}$$

$$\omega_\mu^{03} = \left(0, 0, 0, -\frac{r\dot{a}(t)\sin\theta}{N(t)}\right), \quad \omega_\mu^{12} = \left(0, 0, \sqrt{1-kr^2}, 0\right), \tag{4.13b}$$

$$\omega_\mu^{13} = \left(0, 0, 0, \sqrt{1-kr^2}\,\sin\theta\right), \quad \omega_\mu^{23} = \left(0, 0, 0, \cos\theta\right) \tag{4.13c}$$

where $N(t)$ and $a(t)$ are functions only of the time variable, k is a constant, and $\dot{a}(t) = \frac{\mathrm{d}a(t)}{\mathrm{d}t}$. The choice (4.13) of gauge fields $\omega_\mu^{ab}(x)$ assures that all components of the strength tensor $F_{\mu\nu}^a$ vanish. If we remind the Riemann-Cartan theory of gravitation, then this result implies the vanishing of the torsion tensor $T_{\mu\nu}^\rho = \bar{e}_a^\rho F_{\mu\nu}^a$, in accord with *GR* theory. Here, $\bar{e}_a^\rho$ denotes the inverse of e_μ^a with the properties:

$$e_\mu^a \bar{e}_b^\mu = \delta_b^a, \quad e_\mu^a \bar{e}_a^\nu = \delta_\mu^\nu. \tag{4.14}$$

Using the Eqs. (4.12) and (4.13), we obtain the following expressions of the invariants F, I_1 and I_2 above defined:

$$F = -6\frac{a\ddot{a}N - a\dot{a}\dot{N} + kN^3 + \dot{a}^2 N + 8\lambda^2 a^2 N^3}{a^2 N^3}, \tag{4.15}$$

$$I_1 = -12\frac{kN^2 + \dot{a}^2 + 4\lambda^2 a^2 N^2}{a^2 N^2}, \tag{4.16}$$

and respectively

$$I_2 = 12\frac{\left(kN^3 + \dot{a}^2 N - a\ddot{a}N + a\dot{a}\dot{N}\right)^2}{a^4 N^6}. \tag{4.17}$$

Introducing these expressions into Eq. (4.11) and imposing the variational principle $\delta S_g = 0$ with respect to $N(t)$, $\varphi_1(t)$ and $\varphi_2(t)$, we obtain the corresponding field equations. We write now these equations for the particular case $N(t) = 1$ which is of interest in our model:

$$\begin{cases} -\frac{1}{2}(V_1+V_2)+3H^2(1-2\varphi_1)+3\frac{k}{a^2}(1+2\varphi_1)-2\Lambda = \\ \sqrt{3\left(\dot{\varphi}_2+3H\varphi_2-\frac{k}{Ha^2}\varphi_2\right)} \end{cases} \tag{4.18}$$

$$\frac{k}{a^2}+H^2-\frac{\Lambda}{3}=\frac{V_1'}{12}, \quad V_1'=\frac{\mathrm{d}V_1}{\mathrm{d}\varphi_1}, \quad H=\frac{\dot{a}}{a}, \tag{4.19}$$

$$\dot{H}-\frac{k}{a^2}=-\frac{V_2'}{2\sqrt{3}}, \quad V_2'=\frac{\mathrm{d}V_2}{\mathrm{d}\varphi_2}, \quad \dot{H}=\frac{\mathrm{d}H}{\mathrm{d}t}=\frac{\ddot{a}a-\dot{a}^2}{a^2}, \tag{4.20}$$

where $\Lambda=-12\lambda^2$ is interpreted as cosmological constant [19, 28].

If we consider the limit $\lambda \to 0$, or equivalently $\Lambda = 0$, we obtain the results in Ref. [24]; but, for $\Lambda \neq 0$ we can study in addition the dependence on the cosmological constant of the solutions (without singularities) obtained by solving the Eqs. (4.19)–(4.21). We make also the mention that the Eqs. (4.19) and (4.20) are identically with the constraints (4.5) introduced into the integral of action S_g by means of the Lagrange-multiplier fields $\varphi_1(t)$ and $\varphi_2(t)$.

We can also add matter to the previous model considering the integral of action:

$$S_m=\int \mathrm{d}^4 x e L_m, \tag{4.21}$$

where L_m is the matter density of Lagrangian. In this paper we restrict ourselves to the case without matter. In subsection 4.2 we will obtain a particular solution with fixed cosmological constant Λ = const. Of course, there are possible also solutions with variable cosmological "constant" depending on time. The solution presented below is inspired from the results of Ref. [22] and we show that our cosmological constant Λ is related with the constant H_0 in that work and which is expected to be Planck scale.

It is important to emphasize that in our gauge model of gravitation we do not use a metric, but only the gauge fields (potentials) $e_\mu^a(x)$ and $\omega_\mu^{ab}(x)$ of the gravitational field.

4.3. EXAMPLE OF SOLUTION WITHOUT SINGULARITIES

The solution of Eqs. (4.18)–(4.20) includes a dependence on the cosmological constant Λ. We suppose that the Lagrange-multiplier function $\varphi_1(t)$

is absent, and consider the case when $k = 0$. Then, denoting $\varphi_2(t) = \varphi(t)$ and $V_2(\varphi_2) = V(\varphi)$, the Eqs. (4.18)–(4.20) become:

$$\dot{H} = -\frac{1}{2\sqrt{3}} V', \tag{4.22a}$$

$$\dot{\varphi} = -3H\varphi + \sqrt{3}H - \frac{1}{2\sqrt{3}H} V - \frac{2\Lambda}{\sqrt{3}H}. \tag{4.22b}$$

We consider the potential $V(\varphi)$ of the simple form:

$$V(\varphi) = 2\sqrt{3}\lambda^2 \left(\frac{\varphi^2}{1+\varphi^2} + \frac{24}{\sqrt{3}} \right), \tag{4.23}$$

where λ is the real parameter that determines the cosmological constant Λ. This parameter coincides with the constant H_0 in Ref. [22] that has been interpreted as a Planck scale of the model. Therefore, in our example the Planck scale is related to the cosmological constant Λ. For small values of H and φ, the Eqs. (4.22) can be written as:

$$\dot{H} \simeq -2\lambda^2 \varphi, \tag{4.24a}$$

$$\dot{\varphi}(t) \simeq \frac{\sqrt{3}H^2 - \lambda^2\varphi^2}{H}. \tag{4.24b}$$

These equations have the periodic solution

$$\varphi(t) = \varphi_0 \sin(\omega t), \quad H(t) = \frac{\omega\varphi_0}{2\sqrt{3}} \left[\cos(\omega t) - 1 \right], \tag{4.25}$$

where φ_0 is an integration constant and $\omega = 2 \times 3^{1/4}\lambda$ is the frequency of oscillation of the corresponding gravitational field described by the gauge potentials $e_\mu^a(x)$ and $\omega_\mu^{ab}(x)$. This solution has no singularities and it is valid if the cosmological constant is negative ($\Lambda < 0$). The case with positive cosmological constant ($\Lambda > 0$) can be studied choosing the anti-de-Sitter group as gauge group. But, the deformation parameter λ will be then pure imaginary. We emphasize that there are possible also periodic solutions if we suppose a time-dependent cosmological "constant". In particular, we can consider a cosmological "constant" which is itself a periodic function on time. It will be also of interest to apply the previous method in obtaining non-singular solutions of the gauge theories with internal groups of symmetry.

5. GAUGE FIELD IN INTERACTION WITH ELECTROMAGNETIC FIELD

5.1. ACTION AND FIELD EQUATIONS

Let us suppose now that DS is a gauge group for gravitation. As in previous Section, we introduce 10 gauge fields (the gravitational potentials) $h_\mu^A(x)$, $A = 1, 2, \ldots, 10$, $\mu = 0, 1, 2, 3$, where (x) denotes the local coordinates on M_4, with $x^0 = t$, $x^1 = r$, $x^2 = \theta$, $x^3 = \varphi$. Then we can construct the 2-form $\mathcal{F}$ associated to the potentials $h_\mu^A(x)$ [18]:

$$\mathcal{F} = \frac{1}{2} F_{\mu\nu} \, \mathrm{d}x^\mu \, \mathrm{d}x^\nu, \tag{5.1}$$

where $F_{\mu\nu} = F_{\mu\nu}^A X_A$ are its components which take values in Lie algebra of *DS* group (Lie-algebra valued). The infinitesimal generators X_A of the *DS* group are interpreted as: $X_a \equiv \Pi_a$ (the de-Sitter translation operators) and $X_{[ab]} \equiv M_{ab}$ (the Lorentz transformation operators) with the property $M_{ab} = -M_{ba}$. The constants of structures f_{AB}^C have the expressions given in Eq. (3.1). In fact, the constants (3.1) correspond to a deformation of de-Sitter Lie algebra having λ as parameter [23]. If we consider the contraction $\lambda \to 0$ then the generators Π_a become the generators P_a of space-time translations and the group *DS* contracts therefore to the Poincaré group *P*.

We denote the gravitational gauge fields $h_\mu^A(x)$ by $e_\mu^a(x)$, if $A = a$, and respectively $\omega_\mu^{ab}(x) = -\omega_\mu^{ba}(x)$ if $A = [ab]$. The integral of action associated to the gravitational gauge fields $e_\mu^a(x)$ and $\omega_\mu^{ab}(x)$ will be chosen as [24]:

$$S_g = \frac{1}{16\pi G} \int \mathrm{d}^4 x \, e \, F, \tag{5.2}$$

where $e = \det(e_\mu^a)$ and

$$F = F_{\mu\nu}^{ab} \bar{e}_a^\mu \bar{e}_b^\nu. \tag{5.3}$$

Here, $\bar{e}_a^\mu(x)$ denotes the inverse of $e_\mu^a(x)$ satisfying the usual properties (4.15).

We suppose that the source of the gravitation creates also an electromagnetic field $A_\mu(x)$. The corresponding integral of the action will be chosen in the form [25]:

$$S_{e.m.} = -\frac{1}{4Kg^2}\int \mathrm{d}^4 x\, e\, A^a_\mu \bar{A}^\mu_a, \tag{5.4}$$

with A^a_μ and $\bar{A}^\mu_a$ defined as:

$$A^a_\mu = A^\nu_\mu e^a_\nu, \qquad A^\nu_\mu = \bar{e}^\nu_a \bar{e}^\rho_b \eta^{ab} A_{\mu\rho}, \tag{5.5}$$

and respectively:

$$\bar{A}^\mu_a = A^\nu_\mu \bar{e}^\nu_a, \tag{5.6}$$

and where K is a constant that will be chosen in a convenient form to simplify the solutions of the field equations. Here, $A_{\mu\rho}$ denotes the electromagnetic field tensor:

$$A_{\mu\rho} = \partial_\mu A_\rho - \partial_\rho A_\mu. \tag{5.7}$$

The quantity g is the gauge coupling constant [25, 32]. Then, the total integral of action associated to the system composed of the two fields is given by the sum of the expression (5.2) and (5.4):

$$S = \int \mathrm{d}^4 x \left[\frac{1}{16\pi G} F - \frac{1}{4Kg^2} A^a_\mu \bar{A}^\mu_a \right] e. \tag{5.8}$$

The field equations for the gravitational potentials $e^a_\mu(x)$ can be obtained by imposing the variational principle $\delta_e S = 0$ with respect to $e^a_\mu(x)$. They are [26]:

$$F^a_\mu - \frac{1}{2} F e^a_\mu = 8\pi G T^a_\mu, \tag{5.9}$$

where F^a_μ is defined by:

$$F^a_\mu = F^{ab}_{\mu\nu} \bar{e}^\nu_b, \tag{5.10}$$

and T^a_μ is the energy-momentum tensor of the electromagnetic field [27]:

$$T^a_\mu = \frac{1}{Kg^2}\left(A^b_\mu A^a_\nu \bar{e}^\nu_b - \frac{1}{4} A^b_\nu A^\nu_b e^a_\mu \right). \tag{5.11}$$

The field equations for the other gravitational gauge fields $\omega^{ab}_\mu(x)$ are equivalent with [7, 24]:

$$F^a_{\mu\nu} = 0. \tag{5.12}$$

In the subsection 5.3 we will obtain solution of the gravitational field equations (5.9) and (5.12) supposing that the gravitational fields has spherical symmetry and is created by a point mass m (the source). In the same time, we will admit that the electromagnetic field $A_\mu(x)$ is due to a constant electric charge Q of the same source, *i.e.* we will consider that the point mass m is also electrical charged.

The field equations for the electromagnetic field $A_\mu(x)$ can be obtained also by imposing the variational principle $\delta_A S = 0.$ We will be not write these equations because the expression of the $A_\mu(x)$ are well defined for a fixed point with charge Q.

5.2. MODEL WITH SPHERICAL SYMMETRY

We consider a particular form of spherically gravitational gauge field given by the following ansatz:

$$e_\mu^0 = \left(A, 0, 0, 0\right),\ e_\mu^1 = \left(0, \frac{1}{A}, 0, 0\right),\ e_\mu^2 = \left(0, 0, r, 0\right),\ e_\mu^3 = \left(0, 0, 0, r\sin\theta\right), \tag{5.13}$$

and

$$\begin{gathered} \omega_\mu^{01} = \left(U, 0, 0, 0\right),\ \ \omega_\mu^{02} = \omega_\mu^{03} = 0,\ \ \omega_\mu^{12} = \left(0, 0, A, 0\right), \\ \omega_\mu^{13} = \left(0, 0, 0, A\sin\theta\right),\ \ \omega_\mu^{23} = \left(0, 0, 0, \right)\cos\theta), \end{gathered} \tag{5.14}$$

where A and U are functions only of the $3D$ radius r. We use the above expressions to compute the components of the tensors $F_{\mu\nu}^a$ and $F_{\mu\nu}^{ab}$ defined by the Eqs. (3.2) and (3.3). We obtain the following non-null components of these tensors [32, 30]:

$$F_{10}^0 = \frac{AA' + U}{A}, \tag{5.15}$$

and respectively:

$$\begin{gathered} F_{10}^{01} = U' + 4\lambda^2, \qquad F_{20}^{02} = A\left(U + 4\lambda^2 r\right), \qquad F_{30}^{03} = A\sin\theta\left(U + 4\lambda^2 r\right), \\ F_{21}^{12} = \frac{-AA' + 4\lambda^2 r}{A}, \qquad F_{31}^{13} = \frac{\left(-AA' + 4\lambda^2 r\right)\sin\theta}{A}, \\ F_{32}^{23} = \left(1 - A^2 + 4\lambda^2 r^2\right)\sin\theta, \end{gathered} \tag{5.16}$$

where A' and U' denote the derivatives with respect to variables r.

Using these expressions, we obtain the following expressions for F defined in (5.3) and F_μ^a (only non-null components) given by (5.10):

$$F = -2\frac{r^2 U' + 2rU - 2rAA' + 1 - A^2}{r^2} - 48\lambda^2, \tag{5.17}$$

and respectively:

$$F_0^0 = -\frac{A\left(rU' + 2U + 12\lambda^2 r\right)}{r}, \quad F_1^1 = -\frac{rU' - 2AA' + 12\lambda^2 r}{rA},$$
$$F_2^2 = -\frac{rU - rAA' + 1 - A^2 + 12\lambda^2 r}{r}, \tag{5.18}$$
$$F_3^3 = -\frac{rU - rAA' + 1 - A^2 + 12\lambda^2 r}{r}\sin\theta.$$

The non-null components of the energy-momentum tensor T_μ^a for the electromagnetic field created by the constant charged Q are:

$$T_0^0 = \frac{1}{Kg^2}\frac{AQ^2}{32\pi^2\varepsilon_0^2 r^4}, \qquad T_1^1 = \frac{1}{Kg^2}\frac{Q^2}{32A\pi^2\varepsilon_0^2 r^4}, \tag{5.19a}$$

$$T_2^2 = -\frac{1}{Kg^2}\frac{Q^2}{32\pi^2\varepsilon_0^2 r^3}, \qquad T_3^3 = -\frac{1}{Kg^2}\frac{Q^2}{32\pi^2\varepsilon_0^2 r^3}\sin\theta. \tag{5.19b}$$

Introducing all these results in Eq. (5.9), we obtain the following field equations for $e_\mu^a(x)$:

$$\frac{2AA'}{r} - \frac{1-A^2}{r^2} - 12\lambda^2 + \frac{Q^2}{r^4} = 0, \tag{5.20a}$$

$$U - AA' + rU' + 12\lambda^2 r + \frac{Q^2}{r^4} = 0, \tag{5.20b}$$

$$-2AA' + rU' + 12\lambda^2 r + \frac{Q^2}{r^4} = 0. \tag{5.20c}$$

Here we chosen the constant K in (5.4) so that:

$$\frac{G}{4\pi K g^2 \varepsilon_0^2} = 1. \tag{5.21}$$

Now, if we use the field equations (5.12) for ω_μ^{ab}, then we obtain the following constraint on the component U:

$$U = -AA'. \tag{5.22}$$

Then, the Eqs. (5.20b) and (5.20c) are identically, and these remain two field equations for one unknown function $A(r)$:

$$-\frac{2AA'}{r} + \frac{1-A^2}{r^2} + 12\lambda^2 = \frac{Q^2}{r^4}, \tag{5.23a}$$

$$-\frac{2AA'}{r}+U'+12\lambda^2=-\frac{Q^2}{r^4}. \tag{5.23b}$$

These Eqs. are compatible if we impose the condition:

$$\frac{1-A^2}{r^2}-U'=\frac{2Q^2}{r^4}. \tag{5.24}$$

The solution of this equation can be easy obtained if we use the constraint (5.22) that can be written as:

$$U=-\frac{\left(A^2\right)'}{2}. \tag{5.25}$$

Introducing (5.25) into the Eq. (5.24) and denoting $A^2=y,$ we obtain the following differential equation for the new unknown function $y(r)$:

$$r^2y''-2y+2=\frac{4Q^2}{r^4}. \tag{5.26}$$

The solution of the differential equation (5.26), of second order in the derivatives of the function $y(r)$ with respect to the variable r, is:

$$y\equiv A^2=1+\frac{\alpha}{r}+\frac{Q^2}{r^2}+\beta r^2, \tag{5.27}$$

where α and β are constants of integration. It is known [27, 31] that the constant α is determined by the mass m of the point source that creates the gravitational field:

$$\alpha=-2m. \tag{5.28}$$

The other constant β is determined by condition that the solution (5.27) verify the Eqs. (5.23a)–(5.23b) that now coincide as a consequence of the constraint (5.22). Introducing the solution (5.27) in Eq. (5.23a) we obtain:

$$\beta=4\lambda^2=-\frac{\Lambda}{3},\qquad \Lambda=-12\lambda^2. \tag{5.29}$$

Finally, the solutions of the field equations (5.9) and (5.12) is:

$$A^2=1-\frac{2m}{r}+\frac{Q^2}{r^2}-\frac{\Lambda}{3}r^2,\qquad U=-\frac{\left(A^2\right)'}{2}. \tag{5.30}$$

Here Λ is the cosmological constant of the model [28, 29]. If we consider the contraction $\lambda\to 0,$ then the de-Sitter group becomes the Poincaré group, and the solution (5.30) is known as the Reissner-Nordström.

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