Solitons in dissipative media

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Abstract

A brief overview of recent results in the field of solitons in dissipative systems is given.

One of the most important achievements in nonlinear science has been the discovery of *solitons*, which are self-localized solutions of certain nonlinear partial differential equations describing the evolution of nonlinear dynamical systems with an infinite number of degrees of freedom [1, 2]. The solitons preserve their shape upon interaction and can be viewed as "nonlinear modes" of the corresponding physical system. They are usually attributed to the so called completely integrable models, which are usually obtained as extreme simplifications of complex physical systems. However, solitons or more properly *solitary waves* in nonintegrable systems (either Hamiltonian or dissipative ones) can also be regarded as nonlinear modes, being qualitatively different from their counterparts in completely integrable systems. Thus, the rather complicated behaviour of the system may be described in terms of just a few degrees of freedom. It is important to notice that in nondissipative Hamiltonian systems the solitary waves form as a result of a balance between diffraction and/or dispersion and nonlinearity, whereas in dissipative systems, gain and loss must also be balanced. In the former situation, the solitons form continuous families with one (or few) parameter families, whereas in the latter case the additional balance between gain and loss results in a solitary wave solution having the amplitude and width fixed by the parameters of the governing equations, i.e., soliton solutions are isolated ones. On the other hand, it is quite possible for several isolated soliton solutions to exist for the same equation parameters.

The generic equations which describe dissipative physical systems in a vicinity of a subcritical bifurcation are the complex Ginzburg-Landau (GL) equation, or, in other terms, the nonlinear Schrödinger (NLS) equation perturbed by terms accounting for gain and losses and the Swift-Hohenberg (SH) equation which contains more involved spectral filtering terms [3]-[4] (for recent studies on exact periodic and solitary wave solutions of one-dimensional cubic-quintic SH equations see Ref. [5]). Thus the addition of a fourth-order spectral filtering term into the complex GL equation transforms it into the complex SH equation. Complicated pattern-forming dissipative systems can be described by these complex GL and SH equations [6]. Both equations are qnown to play a ubiquitous role in nonlinear science. They are encountered in several diverse branches of physics, such us for example, in superconductivity, non-equilibrium fluid dynamics and chemical systems, nonlinear optics, Bose-Einstein condensates, and in quantum field theory [3]-[4].

The GL equation exhibits various kind of solutions such as pulselike and shocklike ones, sources, sinks and pulsating (periodic and quasiperiodic) solutions. The GL equation and its different modifications describe various effects in laser physics [7], fluid dynamics [8] and nonlinear optics [9]. The cubic GL equation has been analyzed mainly in the context of plasma physics [10], and its solitary wave solutions are known to be unstable.

In addition to the well investigated continuous GL equation, discrete GL models have also been considered in the past decade. These equations describe Taylor or frustrated vortices in hydrodynamics [11] and semiconductor laser arrays in optics [12].

Complicated optical systems with gain and loss can be described by the complex SH equation. These include synchronously pumped optical parametric oscillators [13], three-level broad-area cascade lasers [14] and large aspect-ratio lasers [15]. The complex SH equation describes also pulse generation in mode-locked lasers with fast saturable absorbers and complicated spectral response [16]. This equation is also used for describing instabilities and pattern formation in the case of Rayleigh-Bénard problem of convection in a horizontal fluid layer in the gravitational field [17], and in oscillating chemical reactions [18].

Later on, the cubic-quintic (CQ) GL equation was put forward, as it admits stable pulselike solutions [19] - [24]. An interesting generalization of the stable pulselike solution of the CQ GL equation in the two-dimensional (2D) case is the possibility of formation of *spinning* solitons having the shape of a ring vortex (i.e., with a hole in the middle). We will see that, on the contrary to a recently found azimuthal instability of spinning doughnut-shaped solitons in the CQ NLS equation, their GL counterparts may be *completely stable*.

A model which has a chance to feature (quasi) stable spinning two-dimensional solitons is that with the CQ nonlinearity, which postulates a nonlinear correction to the medium's refractive index of a specific form [25]. Direct numerical simulations of the dynamics of 2D solitons with $s = 1$ in the CQ model were first reported in Ref. [25] (the stability of 2D solitons with $s = 0$ in the same model was checked by means of direct simulations essentially later [26]). In Ref. [25] it has been concluded that the spinning 2D soliton is fairly robust, provided that its energy was not too small. It was shown to be robust not only against small perturbations, but also against collisions, which were found to be nearly elastic.

As we have said above, complex GL equations constitute a class of universal models describing pattern formation in a great variety of nonlinear dissipative systems [3]. Among the patterns, localized pulses are especially interesting. While the theory of pulses in various 1D models of the GL type was well elaborated [3, 27], much less is known about 2D localized patterns (2D pulses).

In order to make the pulses stable, it is first of all necessary to stabilize its zero background (trivial solution to the GL equation), which can be done within the framework of the CQ GL equation [19] - [24], [28], or a model linearly coupling a cubic GL equation to a linear dissipative one [22]. Although the CQ GL equation was originally introduced by Sergeev and Petviashvili [28] in a 2D form, exactly in order to generate 2D localized pulses in the form of "spiral solitons", much more work has been done to investigate not "solitons" of this type, but rather spiral waves extending to infinity [29, 30]. In particular, the stability of delocalized spiral vortices in a model of a superflow, and interactions of vortices in a two-component GL system have been investigated in Refs. [31] and [32]. On the other hand, nonspiraling axisymmetric 2D patterns in the form of localized "bubbles" were recently studied in other models, e.g., the complex Swift-Hohenberg equation [33].

An analysis of *spiral solitons* of the complex CQ GL equation, i.e., localized 2D objects with an internal vorticity, which, as well as the vortex-ring solitons in conservative models, are characterized by an integer-valued "spin" s, was recentlty performed in Refs. [34]-[35]. As well as in the conservative case, a crucially important issue is the soliton's stability at different values of s.

Note that, in the dissipationless limit, the GL equation goes over into an equation of the NLS type. Accordingly, a spiral soliton turns into a 2D spinning soliton of the type considered above. However, a principal difference is that that the GL equation may only have a few isolated solitary-pulse solutions (normally, two, if one of them is expected to be stable, the second pulse playing the role of an unstable *separatrix* [22]), while its NLS counterpart, as it was demonstrated above, has a continuous family of soliton solutions which may be parametrized by their energy. In the case when all the dissipative parameters of the GL model are small, i.e., it may be considered as a perturbation of its NLS counterpart, isolated pulses that survive as solutions to the GL equation are selected from the continuous family of the NLS solitons by a condition of balance between gain and losses [22].

One may expect that, on the contrary to the conservative NLS equation, in the GL model the vortex ring may be *stable*. Indeed, in the limit when the external size of the ring diverges, the vortex ring turns into a usual delocalized rotating spiral wave [30]. It is known that, generally, the latter wave in dissipative systems has a finite (nonzero) *stability margin* against the spontaneous off-center shift of its inner bubble. On the other hand, in the same limit, the vortex ring in the NLS equation turns into a usual delocalized "optical vortex" (2D dark soliton) [36], which is, obviously, only *neutrally stable* against the spontaneous shift of the inner bubble. Therefore, the interaction of the inner bubble with the outer rim of the large-size but finite NLS soliton may destabilize the whole ring, turning the neutral shift mode into an unstable one, which, as it was explained above, is indeed the case in the CQ NLS equation [37]; however, the above-mentioned stability margin of the inner bubble against the shift inside the GL spiral wave may help to stabilize a *finite-size* spiral soliton too in the GL equation. The simulations show that it is indeed relatively easy to find a spiral soliton which is fairly stable against all the perturbations, including azimuthal ones which are fatal for the NLS vortex ring.

I consider the (2+1)-dimensional CQ complex GL equation in a general form,

$$iA_z + i\delta \cdot A + (1/2 - i\beta)(A_{xx} + A_{yy}) + (1 - i\varepsilon)|A|^2 A - (\nu - i\mu)|A|^4 A = 0. \quad (1)$$

The equation is written in the "optical" notation, assuming evolution along the propagation coordinate z of a beam with the 2D cross section in the plane (x, y). In fact, bulk (3D) optical media is the most appropriate system for experimental generation of vortex rings, see, e.g., Ref. [38]. In that case, $A(x, y; z)$ is the local amplitude of the electromagnetic wave, the diffraction and cubic-self-focusing coefficients are normalized to be 1, ε is the cubic gain, δ and μ are the linear and quintic loss parameters (as a matter of fact, the latter one accounts for the nonlinear gain saturation in optical media), and ν is the quintic self-defocusing coefficient. Lastly, β is an effective diffusion coefficient (in optical media, diffusion takes place if light creates free charge carriers, which may take place, e.g., in semiconductor waveguides).

The spiral solitons are axisymmetric solutions to Eq. (1) of the form $A(x, y; z) = U(z, r)\exp(is\theta)$, where r and θ are polar coordinates in the (x, y) plane, and s is the above-mentioned integer "spin" (topological charge of the vortex). The complex amplitude $U(z, r)$ obeys an equation

$$iU_z + i\delta \cdot U + (1/2 - i\beta)(U_{rr} + r^{-1}U_r - s^2 r^{-2} U) + (1 - i\varepsilon)|U|^2 U - (\nu - i\mu)|U|^4 U = 0, \quad (2)$$

which is supplemented by boundary conditions stating that $U \sim r^s$ at $r \to 0$, and $U(r)$ decays exponentially at $r \to \infty$. Note that the localized solution can be interpreted as a *spiral soliton* because the function $U(r)$ is complex, $U(r) \equiv |U(r)| \exp(i\Phi(r))$, hence equal-phase curves $s\theta + \Phi(r) =$ const are spirals, rather than straight lines $s\theta =$ const, as in the case of the CQ NLS equation, where $U(r)$ is real.

The first purpose is to find stationary localized solutions to Eq. (2) which must be stable within the framework of this equation, i.e., they must be *attractors*, similarly to stable solitons in various forms of the 1D GL equation [20] - [24]. As the stability of the spiral solitons against the most dangerous azimuthal perturbations is not comprised by Eq. (2), it will be considered below separately.

In order to find the solutions, we have performed numerical simulations of Eq. (2) at many different values of parameters and using various initial configurations $U(r; z = 0)$.

The spinning solitons, as well as the nonspinning ones, are found to be strong attractors, as they can be generated from a large variety of inputs. Another important finding is that stable solitons with different values of the vorticity considered here, $s = 0, 1, 2$, *coexist* in a large domain of the parameter space. In other words, each soliton is a strong attractor inside its own class of pulses, distinguished by the value of the spin, which plays the role of a *topological invariant*.

An important characteristic of solitons is their *integral power* (which is sometimes also called "energy", or "intensity"). It is defined in the usual way, as $I = 2\pi \int_0^{+\infty} |U(r)|^2 rdr$. The power has been found to take very different values for the coexisting solitons with different values of the spin.

In some regions of the parameter space we have found apparently stable nonspinning and spinning pulsating solutions (*breathers*), which demonstrate persistent quasiperiodic internal vibrations, similar to those found in the 1D version of CQ GL equation [23, 24]. These robust pulsating solitons are a characteristic feature of dissipative physical systems and they do not have anything in common with the superposition of zero-velocity solitons of the NLS equation which produces pulsating solutions. I mention also that the nonspiraling ($s = 0$) and spiraling ($s = 1, 2$) breathers coexist with each other at the same values of the parameters, whereas the stationary (nonvibrating) solitons do *not* exist at these parameter values.

As it was explained above, the comparison with the spinning solitons in the CQ NLS model strongly suggests that the overall stability of the solitons is determined by azimuthal perturbations breaking the axial symmetry of the solutions. Recall that the above equation (2) cannot include azimuthal perturbations. Therefore, in order to test this kind of the stability, we have simulated the full equation (1) directly in the Cartesian coordinates. Doing this, we have found that both nonspinning and spiral solitons found above from Eq. (2) are remarkably *stable* against azimuthal perturbations, in accordance with the qualitative arguments presented before.

Although large-scale simulations of the full underlying equation are quite sufficient to predict observation of stable solitons in experiments, this does not provide for a mathematically complete evidence of the solitons' stability. As well as in the case of the solitons in conservative media, a direct way to prove the true stability is to consider the eigenvalue problem produced by the linearization of the evolution equation around the corresponding stationary solutions. It was performed this analysis too, aiming to compute the largest growth rates (eigenvalues) of the azimuthal perturbation eigenmodes for different values of the perturbation index n. In *all the cases* when the direct simulations produced apparently stable stationary solitons, we have found that the largest growth rate of the perturbation eigenmodes has a *negative* real part, thus corroborating the stability of the corresponding stationary solitons.

In order to assess the range in the space of the initial configurations for which the stable solitons are attractors, we have additionally performed a large number of direct simulations with initial pulses in the form of Gaussians with intrinsic vorticity. Simulations of this kind are necessary because, in a real experiment, input pulses normally have the Gaussian shape (the vorticity can easily be lent to a pulse by passing it through a phase mask [38]). In all the cases in which the existence of stable spiral solitons was known, the initial Gaussians rapidly developed into them, keeping the initial value of the spin.

A challenging problem is the interaction between spiral solitons with equal or different vorticities (interactions between spirals extending to infinity have been already studied in detail by means of analytical and numerical methods [30]). If the solitons are far separated, and their vorticities are equal or opposite, the interaction can be described analytically in terms of an effective potential, using a technique elaborated in Ref. [39]. However, in the most interesting case when the solitons are essentially overlapped and, hence, they interact strongly, heavy direct simulations are necessary.

It was found that the low energy spinning solitons in the conservative model are always subject to an instability breaking their axial symmetry and leading to splitting of the soliton into several fragments flying out in tangential directions, each one being a (perfectly stable) zero-spin soliton. On the contrary to this, the spiral solitons in the two-dimensional CQ GL equation may be fully stable, and they may coexist at different values of the spin. Moreover, the two-dimensional CQ GL equation also gives rise to stable nonspinning and spiral pulsating solitons (breathers), which coexist with each other, but do not coexist with the nonvibrating solitons.

Recently three novel varieties of spiraling and nonspiraling axisymmetric solitons in the complex GL equation have been found [35]. These are irregularly "erupting" pulses, and two differnt types of very broad stationary ones, found near a border between ordinary pulses and expanding fronts. The spiraling erupting solitons were found to be unstable against azimuthal perturbations. However, the "flat -top" solitons were found to be stable, whereas the "composite" solitons, that unlike the "flat-top" ones, have some weakly pronounced intrinsic structure, have found to be unstable [35].

The formation of localized structures (solitons) and various kind of patterns in nonlinear optics have received a great deal of attention in the recent years, from both the theoretical and experimental point of view [40] - [47]. Several kinds of configurations have been considererd: unidirectional propagation, counterpropagating beams, systems with single feedback miror, nonlinear optical media such as local and instantaneous cubic (Kerr) media or quadratically nonlinear media contained in optical cavities with planar or spherical mirrors, and arrays of lasers. Planar cavities (resonators) filled with nonlinear optical media are the prototype configurations and due to their intrinsic feedback, they exhibit dynamical instabilities leading to fundamental spatio-temporal phenomena such as bistability, chaos, self-pulsations, or optical pattern formation. From the experimental point of view these cavities provide a considerable optical field enhancement inside resonators, thus reducing the optical power requirements for the observation of the above-mentioned fundamental effects.

Though the research has focused in the past years on the case of cubic nonlinearities, in recent years the interest has switched to quadratically nonlinear media in the intracavity geometry, either for optical parametric oscillators (OPOs) [48] - [52] or for second-harmonic generation (SHG) [53] - [55].

In the majorities of theoretical studies, the mean field approximation as well as the assumption of an instantaneous response of the medium have been used. However, very recently the assumption of the mean field has been relaxed, and two-dimensional localized structures were shown to form in a passive ring cavity filled with either a two-level medium or with a quadratically nonlinear medium in a degenerate OPO scheme [56] - [60]. We mention that recently in Ref. [59], for the case of a degenerate OPO, the equations of the mean-field model have been derived from those of the propagation model and a quantitative comparison between the two models has been performed. Very recently, mean-fieeld equations for intra-cavity second-harmonic generation (the up-conversion case) have been derived [61]. Based on this mean-field model, a variety of quadratic cavity solitons was shown to exist, in particular the existence of triangular cavity solitons has been revealed.

Spatial solitons can exist in nonlinear optical resonators with and without amplification. In the last decade different kinds of these localized structures such as vortex solitons, bright or dark solitons, and phase solitons have been experimentally discovered. Recently, more complex soliton structures in the form of "soliton clusters" have been also theoretically investigated in active systems such as driven optical microcavities [62]. As concerns the observation of these stable localized states, which can be considered as "bits" of information for applications in parallel information storage and processing, we mention recent experimental studies on pattern formation and localized structures in OPOs [63] - [64] and in semiconductor microresonators [65] - [66] (for a recent comprehensive review see, e.g., Ref. [67]).

To conclude, I would like to stress that after several years of theoretical and experimental progress in the field of solitons in dissipative media and, in particular, in optical microresonators, many frontiers in this area still remain to be explored. However, who knows how our lives will change when data processing with optical spatial (or spatiotemporal) solitons as elementary bits of information finally reach maturity.

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